

Navier-Stokes Equations and Forward-Backward Stochastic Differential Systems *

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March 22, 2013

Abstract

In the paper, we consider a special coupled forward-backward stochastic differential system (FBSDS) which is associated to the viscous incompressible Navier-Stokes equation and provides a probabilistic solution to the latter via the Feynman-Kac formula. With a probabilistic method, we first prove the existence and uniqueness of the solution to the FBSDS. Then under the same conditions, we verify that this solution leads to a unique local strong solution to the associated Navier-Stokes equation, and for both cases of the small Reynolds number and dimension two, we further give the global strong solutions. Our probabilistic representation formula for the solution to the Navier-Stokes equation involves neither integral nor gradient operators, and it can serve to generate a Monte Carlo solution to the viscous incompressible Navier-Stokes equation.

Keywords: forward-backward stochastic differential system, Navier-Stokes equation, Feynman-Kac formula.

1 Introduction

The standard deterministic Navier-Stokes equation describes the evolution of the velocity field of an incompressible, viscous fluid moving in a domain of \mathbb{R}^d ($d = 2$ or 3 throughout this work), and takes the following form:

$$\begin{cases} \partial_t u - \frac{\nu}{2} \Delta u + (u \cdot \nabla)u + \nabla p + \bar{f} = 0, & t \geq 0; \\ \nabla \cdot u = 0, & u(0) = u_0, \end{cases} \quad (1.1)$$

where u is the d -dimensional velocity field of a fluid, p is the pressure field, $\nu \in (0, \infty)$ is the viscosity coefficient, and \bar{f} is the external force which, without any loss of generality, is taken to be divergence free. Let $T \in (0, \infty)$ be a real sufficiently big number. If (u, p) solves the initial Cauchy problem (1.1), then (\tilde{u}, \tilde{p}) defined by the following time-reversing transformation

$$\tilde{u}(t, x) = -u(T - t, x), \quad \tilde{p}(t, x) = p(T - t, x), \quad f(t, x) = \bar{f}(T - t, x), \quad \text{for } t \leq T,$$

solves the following *terminal* Cauchy problem:

$$\begin{cases} \partial_t \tilde{u} + \frac{\nu}{2} \Delta \tilde{u} + (\tilde{u} \cdot \nabla) \tilde{u} + \nabla \tilde{p} + f = 0, & t \leq T; \\ \nabla \cdot \tilde{u} = 0, & \tilde{u}(T) = G := -u_0, \end{cases} \quad (1.2)$$

which is also called (*backward*) Navier-Stokes equation due to its equivalence to the former.

*This research is supported by the Natural Science Foundation of China (Grants #10325101 and #11171076), the Science Foundation for Ministry of Education of China (No.20090071110001), and the Chang Jiang Scholars Programme. Part of the work was done when the second author visited Department of Mathematics, ETH, Zürich in the summer of 2011. The hospitality of ETH is greatly appreciated. He also would like to thank Professors Michael Struwe and Alain-Sol Sznitman for very helpful discussions and comments related to Navier-Stokes equations. *E-mail:* delbaen@math.ethz.ch (Freddy Delbaen), qiujiann@gmail.com (Jinniao Qiu), sjtang@fudan.edu.cn (Shanjian Tang).

It is extremely hard to completely solve the Navier-Stokes equation (1.2) or (1.1), and it still remains to be a striking important open problem. Charles Fefferman in his well-known article [17] finally commented, “Standard methods from PDE appear inadequate to settle the problem. Instead, we probably need some deep, new ideas.” In the paper, we develop a probabilistic methodology to study the Navier-Stokes equation. More precisely, the Navier-Stokes equation (1.2) is associated to the following coupled forward-backward stochastic differential system (FBSDS):

$$\left\{ \begin{array}{l} dX_s(t, x) = Y_s(t, x) ds + \sqrt{\nu} dW_s, \quad s \in [t, T]; \\ X_t(t, x) = x; \\ -dY_s(t, x) = \left[f(s, X_s(t, x)) + \tilde{Y}_0(s, X_s(t, x)) \right] ds - \sqrt{\nu} Z_s(t, x) dW_s, \quad s \in [t, T]; \\ Y_T(t, x) = G(X_T(t, x)); \\ -d\tilde{Y}_s(t, x) = \sum_{i,j=1}^d \frac{27}{2s^3} Y_t^i Y_t^j(t, x + B_s) \left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i \right) \left(B_s^j - B_{\frac{2s}{3}}^j \right) B_{\frac{s}{3}} ds \\ \quad - dM_s, \quad s \in (0, \infty); \\ \tilde{Y}_\infty(t, x) = 0. \end{array} \right. \quad (1.3)$$

Here and in the following, B and W are two independent d -dimensional standard Brownian motions. The drift part of $\{Y_s(t, x), s \in [t, T]\}$ (see the third equality of FBSDS (1.3)) at time s depends on \tilde{Y}_0 , and that of $\{\tilde{Y}_s(t, x), s \in [t, T]\}$ depends on $Y_t(t, x + B_s)$, which make our system (1.3) strikingly different from the conventional coupled forward-backward stochastic differential equations (FBSDEs) (see [1, 20, 29, 34, 37, 35, 43]). Furthermore, both backward stochastic differential equations (BSDEs) in FBSDS (1.3) are defined on two different time-horizons $[t, T]$ and $(0, \infty)$. The H^m -solution ($m > d/2$, see Definition 3.2) (X, Y, Z, \tilde{Y}_0) to FBSDS (1.3) will be connected to the strong solution (\tilde{u}, \tilde{p}) to the Navier-Stokes equation (1.2) in the following manner:

$$Y_s(t, x) = \tilde{u}(s, X_s(t, x)), \quad Z_s(t, x) = \nabla \tilde{u}(s, X_s(t, x)), \quad \text{and} \quad \tilde{Y}_0(t, x) = \nabla \tilde{p}(t, x), \quad (s, x) \in [t, T] \times \mathbb{R}^d.$$

FBSDEs have been connected to a system of semi-linear parabolic partial differential equations (PDEs) (see among many others [1, 20, 29, 34, 35, 43]). However, Navier-Stokes equation (1.2) usually goes beyond that context, as it has the nonlocal constraint $\nabla \cdot \tilde{u} = 0$. Our difficulty is two-fold: one is nonlinearity and the other is the nonlocal constraint. To attack the divergence-free constraint, we introduce the last BSDE in the infinite time interval. To concentrate our attention on the primary connection between FBSDS (1.3) and Navier-Stokes equation (1.2), FBSDS (1.3) is discussed from an analytic point of view only in a Hilbert space, though it might be addressed in other spaces like Hölder space.

There is a long history on the formalisms to represent solutions of PDEs as the expected functionals of stochastic processes. We only mention here those concerning a deterministic incompressible Navier-Stokes equation. The velocity field is related to the vorticity field in a linear fashion by the Biot-Savart law, and moreover, the analysis of the vorticity field is fundamental to the issues like the possible emergence of singularities (for instance, see [3, 30]). For the two-dimensional case, since the vorticity obeys a Fokker-Planck type parabolic PDE, the random vortex method was formulated by Chorin [8] who used random walks and a particle limit to represent the vorticity field, and Busnello [6] used the Girsanov transformation to give a probabilistic representation of the vorticity field. The latter work was further extended by Busnello, Flandoli and Romito [7] to the three-dimensional case, where the vorticity field turns out to satisfy a parabolic PDE with an additional stretching term. As noted by Busnello, Flandoli and Romito [7], a partially similar representation formula for the vorticity field of the three-dimensional Navier-Stokes equations had been given before by Esposito et al. [16, 15] but without the probabilistic representation for the Biot-Savart law. Note that a probabilistic interpretation for the Biot-Savart law was given by Busnello, Flandoli and Romito [7] and Busnello [6], where the Bismut-Elworthy-Li formula is used so that the velocity can be recovered from the vorticity through probabilistic approaches. Le Jan and Sznitman [24] interpreted the Fourier transformation of the Laplacian of the three-dimensional

velocity field in terms of a backward branching process and a composition rule along the associated tree, and got a new existence theorem, and their approach was extensively studied and generalized by others (see, for instance [4, 32]).

Recently, Constantin and Iyer [9, 10] and Iyer [21, 22, 23] derived a stochastic representation for the incompressible Navier-Stokes equations based on stochastic Lagrangian paths and gave a self-contained proof of the existence. Later, Zhang [44] considered a backward analogue and provided short elegant proofs for the classical existence results. However, their probabilistic representations were not so complete as those of [4, 6, 7, 24, 32] in the sense that they retain the Leray-Hodge projection \mathbf{P} and the gradient operator (see [9, 10, 21, 22, 23, 44]). Ours involves neither integral nor gradient operators, and it can serve to generate a Monte Carlo solution to the Navier-Stokes equation. We also mention that Cruzeiro and Shamaraova [11] established a connection between the strong solution to the spatially periodic Navier-Stokes equations and a solution to a system of FBSDEs on the group of volume-preserving diffeomorphisms of a flat torus, and that Qiu, Tang and You [38] considered a similar non-Markovian FBSDEs to ours (1.3) in the two-dimensional spatially periodic case, and studied the well-posedness of the corresponding backward stochastic PDEs. The list of literature on probabilistic approaches to the Navier-Stokes equations is not exhausted here, and there are many others.

The rest of this paper is organized as follows. In Section 2, we introduce notations and functional spaces, and recall auxiliary results. In Section 3, a lemma on equivalent norms and the definition of the solution to FBSDEs (1.3) are given first and then the FBSDEs is connected to the Navier-Stokes Equation, which constitutes our main result of Theorem 3.3. In Section 4, we prove Theorem 3.3. In Section 5, we give global existence of the solution for the small Reynolds number and the two-dimensional cases, respectively. Finally in Section 6 as an appendix, we prove Lemmas 3.1 and 4.1.

2 Preliminaries

Let $(\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space on which are defined two d -dimensional standard Brownian motion $W = \{W_t : t \in [0, \infty)\}$ and $B = \{B_t : t \in [0, \infty)\}$ such that $\{\tilde{\mathcal{F}}_t\}_{t \geq 0}$ is the natural filtration generated by W and B , and augmented by all the \mathbb{P} -null sets in $\tilde{\mathcal{F}}$. By $\{\mathcal{F}_t\}_{t \geq 0}$ and $\{\mathcal{F}^B_t\}_{t \geq 0}$, we denote the natural filtration generated by W and B respectively, and they are both augmented by all the \mathbb{P} -null sets. \mathcal{P} is the σ -Algebra of the predictable sets on $\Omega \times [0, T]$ associated with $\{\mathcal{F}_t\}_{t \geq 0}$.

Denote by $|\cdot|$ (respectively, $\langle \cdot, \cdot \rangle$ or \cdot) the norm (respectively, scalar product) in finite-dimensional Hilbert space such as $\mathbb{R}, \mathbb{R}^k, \mathbb{R}^{k \times l}$ where k, l are positive integers and

$$|x| := \left(\sum_{i=1}^k x_i^2 \right)^{1/2} \quad \text{and} \quad |y| := \left(\sum_{i=1}^k \sum_{j=1}^l y_{ij}^2 \right)^{1/2} \quad \text{for } (x, y) \in \mathbb{R}^k \times \mathbb{R}^{k \times l}.$$

For each Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and real $q \in [1, \infty]$, we denote by $S^q([t, \tau]; \mathcal{X})$ the set of \mathcal{X} -valued, \mathcal{F}_t -adapted and càdlàg processes $\{X_s\}_{s \in [t, \tau]}$ such that

$$\|X\|_{S^q([t, \tau]; \mathcal{X})} := E \left[\sup_{s \in [t, \tau]} \|X_s\|_{\mathcal{X}}^q \right]^{1/q} < \infty.$$

$L^q_{\mathcal{F}}(t, \tau; \mathcal{X})$ denotes the set of (equivalent classes of) \mathcal{X} -valued predictable processes $\{X_s\}_{s \in [t, \tau]}$ such that

$$\|X\|_{L^q_{\mathcal{F}}(t, \tau; \mathcal{X})} := E \left[\int_t^\tau \|X_s\|_{\mathcal{X}}^q ds \right]^{1/q} < \infty.$$

Both $(S^q([t, \tau]; \mathcal{X}), \|\cdot\|_{S^q([t, \tau]; \mathcal{X})})$ and $(L^q_{\mathcal{F}}(t, \tau; \mathcal{X}), \|\cdot\|_{L^q_{\mathcal{F}}(t, \tau; \mathcal{X})})$ are Banach spaces.

Define the set of multi-indices

$$\mathcal{A} := \{\alpha = (\alpha_1, \dots, \alpha_d) : \alpha_1, \dots, \alpha_d \text{ are nonnegative integers}\}.$$

For any $\alpha \in \mathcal{A}$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, denote

$$|\alpha| = \sum_{i=1}^d \alpha_i, \quad x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}, \quad D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}}.$$

For differentiable transformations ϕ, ψ on \mathbb{R}^d , define the Jacobi matrix $\nabla \phi$ of ϕ :

$$\nabla \phi = \begin{pmatrix} \partial_{x_1} \phi^1, \partial_{x_2} \phi^1, \dots, \partial_{x_d} \phi^1 \\ \partial_{x_1} \phi^2, \partial_{x_2} \phi^2, \dots, \partial_{x_d} \phi^2 \\ \dots, \dots, \dots, \dots \\ \partial_{x_1} \phi^d, \partial_{x_2} \phi^d, \dots, \partial_{x_d} \phi^d \end{pmatrix} \quad (2.1)$$

whose transpose is denoted by $\nabla^t \phi$, the divergence $\operatorname{div} \phi = \nabla \cdot \phi$, and the matrix

$$\phi \otimes \psi = \begin{pmatrix} \phi^1 \psi^1, \phi^1 \psi^2, \dots, \phi^1 \psi^d \\ \phi^2 \psi^1, \phi^2 \psi^2, \dots, \phi^2 \psi^d \\ \dots, \dots, \dots, \dots \\ \phi^d \psi^1, \phi^d \psi^2, \dots, \phi^d \psi^d \end{pmatrix}. \quad (2.2)$$

Now we extend several spaces of real-valued functions to those of vector-valued functions. For a positive integer number l and k , we denote by $C_c^\infty(\mathbb{R}^l; \mathbb{R}^k)$ (respectively, $C_c^\infty(\mathcal{O}; \mathbb{R}^k)$ for each open set $\mathcal{O} \subset \mathbb{R}^l$) the set of all infinitely differentiable \mathbb{R}^k -valued functions with compact supports on \mathbb{R}^l (\mathcal{O} , respectively) and by $\mathcal{D}'(\mathbb{R}^l; \mathbb{R}^k)$ the totality of all the \mathbb{R}^k -valued general functions with each component being Schwartz distribution. For simplicity, we write C_c^∞ and \mathcal{D}' for the case $l = k = d$. On \mathbb{R}^d we denote by \mathcal{S} (\mathcal{S}' , respectively) the set of all the \mathbb{R}^d -valued functions whose elements are Schwartz functions (tempered distributions, respectively). We shall denote by (\cdot, \cdot) not only the duality between C_c^∞ and \mathcal{D}' but also the duality between \mathcal{S} and \mathcal{S}' . Then the Fourier transform $\mathcal{F}(f)$ of $f \in \mathcal{S}$ is given by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(-\sqrt{-1}\langle x, \xi \rangle) f(x) dx, \quad \xi \in \mathbb{R}^d,$$

and the inverse Fourier transform $\mathcal{F}^{-1}(f)$ is given by

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(\sqrt{-1}\langle x, \xi \rangle) f(\xi) d\xi, \quad x \in \mathbb{R}^d.$$

Extended to the general function space \mathcal{S}' , the Fourier transform defines an isomorphism from \mathcal{S}' onto itself. As usual, for each $s \in \mathbb{R}$ and $f \in \mathcal{S}'$, we denote the Bessel potential $I_s(f) := (1 - \Delta)^{s/2} f = \mathcal{F}^{-1}((1 + |\xi|^2)^{s/2} \mathcal{F}(f)(\xi))$.

For each positive integer l , $1 \leq q \leq \infty$ and $m = 0, 1, \dots$ by $L^q(\mathbb{R}^l)$ and $H^{m,q}(\mathbb{R}^l)$ (L^q and $H^{m,q}$, with a little notional abuse), we denote the usual \mathbb{R}^l -valued Lebesgue and Sobolev spaces on \mathbb{R}^d , respectively. $H^{m,q}$ is equipped with the norm:

$$\|\phi\|_{m,q} := \begin{cases} \left(\|\phi\|_{L^q}^q + \sum_{|\alpha|=1}^m \|D^\alpha \phi\|_{L^q}^q \right)^{1/q}, & \phi \in H^{m,q}, q \in [1, \infty); \\ \|\phi\|_{L^\infty} + \sum_{|\alpha|=1}^m \|D^\alpha \phi\|_{L^\infty}, & \phi \in H^{m,\infty}, \end{cases}$$

which is equivalent to the norm:

$$\|\phi\|_{m,q} := \|(1 - \Delta)^{\frac{m}{2}} \phi\|_{L^q}, \quad \phi \in H^{m,q}, \quad \text{for } q \in (1, \infty).$$

Both norms will not be distinguished unless there is a confusion. Furthermore, using the Bessel potentials, we define the Sobolev space $H^{m,q} := I_{-m}(L^q)$ for $m \in \mathbb{Z} \setminus (0 \cup \mathbb{N})$ and $q \in (1, \infty)$. In particular, for the case of $q = 2$, $H^{m,2}$ is a Hilbert space with the inner product:

$$\langle \phi, \psi \rangle_m := \int_{\mathbb{R}^d} \langle I_{m/2} \phi(x), I_{m/2} \psi(x) \rangle dx, \quad \phi, \psi \in H^{m,2}.$$

We define the duality between $H^{s,q}$ and $H^{r,q'}$ for $q \in (1, \infty)$ and $q' = q/(q-1)$ as:

$$\langle \phi, \psi \rangle_{s,r} := \int_{\mathbb{R}^d} \langle I_{s/2} \phi(x), I_{r/2} \psi(x) \rangle dx, \quad \phi \in H^{s,q}, \psi \in H^{r,q'}.$$

For simplicity, we write the space H^m and the norm $\|\cdot\|_m$ for $H^{m,2}$ and $\|\cdot\|_{m,2}$, respectively.

Define

$$\mathcal{D}_\sigma := \{\phi \in C_c^\infty : \nabla \cdot \phi = 0\}.$$

Denote by $H_\sigma^{m,q}$ the completion of \mathcal{D}_σ under the norm $\|\cdot\|_{m,q}$, which is a Banach subspace of $H^{m,q}$.

Now we introduce several spaces of continuous functions. For each positive integer l , nonnegative integer k and domain $\mathcal{O} \subset \mathbb{R}^d$, we denote by $C(\mathcal{O}, \mathbb{R}^l)$, $C^k(\mathcal{O}, \mathbb{R}^l)$ and $C^{k,\delta}(\mathcal{O}, \mathbb{R}^l)$ with $\delta \in (0, 1)$ the continuous function spaces equipped with the following norms respectively:

$$\begin{aligned} \|\phi\|_{C(\mathcal{O}, \mathbb{R}^l)} &:= \sup_{x \in \mathcal{O}} |\phi(x)|, \quad \|\phi\|_{C^k(\mathcal{O}, \mathbb{R}^l)} := \|\phi\|_{C(\mathcal{O}, \mathbb{R}^l)} + \sum_{|\alpha|=1}^k \|D^\alpha \phi\|_{C(\mathcal{O}, \mathbb{R}^l)}, \\ \|\phi\|_{C^{k,\delta}(\mathcal{O}, \mathbb{R}^l)} &:= \|\phi\|_{C^k(\mathcal{O}, \mathbb{R}^l)} + \sum_{|\alpha|=k} \sup_{x,y \in \mathcal{O}, x \neq y} \frac{|D^\alpha \phi(x) - D^\alpha \phi(y)|}{|x-y|^\delta}, \end{aligned}$$

with the convention that $C^0(\mathcal{O}, \mathbb{R}^l) \equiv C(\mathcal{O}, \mathbb{R}^l)$. Whenever there is no confusion, we write $C(\mathbb{R}^d)$, C^k , and $C^{k,\delta}$ for $C(\mathbb{R}^d, \mathbb{R}^l)$, $C^k(\mathbb{R}^d, \mathbb{R}^l)$ and $C^{k,\delta}(\mathbb{R}^d, \mathbb{R}^l)$, respectively. We define $C^\infty(\mathbb{R}^d) := \cap_{k=1}^\infty C^k(\mathbb{R}^d)$.

In an obvious way, we define spaces of Banach space valued functions such as $C(0, T; H^{m,q})$ and $L^r(0, T; H^{m,q})$ for $m \in \mathbb{Z}$, $r, q \in [1, \infty]$, and related local spaces like the following ones:

$$L_{\text{loc}}^r(T_0, T; H^{m,q}) := \bigcup_{T_1 \in (T_0, T]} L^r(T_1, T; H^{m,q}), \quad C_{\text{loc}}((T_0, T]; H^{m,q}) := \bigcup_{T_1 \in (T_0, T]} C([T_1, T]; H^{m,q}).$$

We have the following properties on Sobolev spaces, whose proof is omitted.

Lemma 2.1. *There holds the following assertions:*

(i) *the space H^n , $n > d/2 + k$, $k \in \mathbb{Z}^+ \cup 0$, is continuously embedded into the space $C^{k,\delta}$ for any $\delta \in (0, (n - d/2 - k) \wedge 1)$, i.e., there exists a constant $C > 0$ such that*

$$\|\phi\|_{C^{k,\delta}} \leq C \|\phi\|_n, \quad \forall \phi \in H^n;$$

(ii) *if $1 < r < s < \infty$ and $m, n \in \mathbb{Z}$ satisfying $\frac{d}{s} - m = \frac{d}{r} - n$, then $H^{n,r}$ is continuously embedded into $H^{m,s}$, i.e., there exists a constant $C > 0$ such that*

$$\|\phi\|_{m,s} \leq C \|\phi\|_{n,r}, \quad \forall \phi \in H^{n,r};$$

(iii) *for $m \in \mathbb{Z}^+ \cup \{0\}$, there exists a constant $C > 0$ such that, for any $\phi, \psi \in L^\infty \cap H^m$,*

$$\begin{aligned} \|\phi\psi\|_m &\leq C \{ \|\phi\|_{L^\infty} \|D^m \psi\|_{L^2} + \|\psi\|_{L^\infty} \|D^m \phi\|_{L^2} \}, \\ \sum_{0 \leq |\alpha| \leq m} \|D^\alpha(\phi\psi) - \phi D^\alpha \psi\|_{L^2} &\leq C \{ \|\nabla \phi\|_{L^\infty} \|D^{m-1} \psi\|_{L^2} + \|D^m \phi\|_{L^2} \|\psi\|_{L^\infty} \}, \quad m \geq 1; \end{aligned}$$

(iv) *for any $s > d/2$, H^s is a Banach algebra, i.e., there exists a constant $C > 0$ such that,*

$$\|\phi\psi\|_s \leq C \|\phi\|_s \|\psi\|_s, \quad \forall \phi, \psi \in H^s.$$

The first two assertions of Lemma 2.1 are borrowed from the well-known embedding theorem in Sobolev space theory (see [42]) and the others are referred to [30, Lemma 3.4, Page 98]. For simplicity, we shall denote by \hookrightarrow the embedding relationship, i.e., by $A \hookrightarrow B$ we mean that normed space $(A, \|\cdot\|_A)$ is embedded into $(B, \|\cdot\|_B)$ with a constant C such that

$$\|f\|_B \leq C \|f\|_A, \quad \forall f \in A.$$

Remark 2.1. Note that $d = 2$ or 3 throughout this work. For any $h, g \in H^2$, we have

$$\begin{aligned}
& \|h \cdot g\|_1^2 \\
&= \|h \cdot g\|_0^2 + \sum_{i=1}^d \|\partial_{x_i} h \cdot g\|_0^2 + \sum_{i=1}^d \|h \cdot \partial_{x_i} g\|_0^2 \\
&\leq C \left\{ \|h\|_{C(\mathbb{R}^d)}^2 \|g\|_0^2 + \|\nabla h\|_0^{2\beta} \|\nabla h\|_1^{2-2\beta} \|g\|_0^{2\beta} \|g\|_1^{2-2\beta} + \|h\|_{C(\mathbb{R}^d)}^2 \|g\|_1^2 \right\} \\
&\quad (\text{ using Gagliardo-Nirenberg Inequality (see [18, 28, 31])}) \\
&\leq C \|h\|_2^2 \|g\|_1^2,
\end{aligned}$$

where $\beta := 1 - d/4$, and H^2 is embedded into $C^{0,\delta}$ for some $\delta \in (0, 1)$. In view of Lemma 2.1, we have for any integer $m > d/2$,

$$\|hg\|_{m-1} \leq C(m, d) \|h\|_m \|g\|_{m-1}, \quad \forall h \in H^m, g \in H^{m-1}.$$

Lemma 2.2. *There hold the following :*

$$|\langle (\phi \cdot \nabla) \psi, \psi \rangle_{m-1, m+1}| \leq C \|\phi\|_m \|\psi\|_m^2, \quad m \geq 2, \phi \in H_\sigma^m, \psi \in H_\sigma^{m+1}. \quad (2.3)$$

and

$$|\langle (\phi \cdot \nabla) \phi, \phi \rangle_m| \leq C \|\nabla \phi\|_{L^\infty(\mathbb{R}^d)} \|\phi\|_m^2, \quad m \geq 2, \phi \in H_\sigma^{m+1}. \quad (2.4)$$

Proof. The first inequality (2.3) is referred to [26]. For the reader's convenience, we give a simple proof of the second inequality (2.4).

For any multi-index α such that $|\alpha| \leq m$, we have

$$D^\alpha((\phi \cdot \nabla) \phi) = (\phi \cdot \nabla) D^\alpha \phi + \sum_{0 < \beta \leq \alpha} C(\alpha, \beta) (D^\beta \phi \cdot \nabla) D^{\alpha-\beta} \phi.$$

Since $\langle (\phi \cdot \nabla) D^\alpha \phi, D^\alpha \phi \rangle_0 = 0$, we have from Assertion (iii) of Lemma 2.1 that

$$\begin{aligned}
\langle (\phi \cdot \nabla) \phi, \phi \rangle_m &= \sum_{0 < |\alpha| \leq m} \sum_{0 < \beta \leq \alpha} C(\alpha, \beta) \langle (D^\beta \phi \cdot \nabla) D^{\alpha-\beta} \phi, D^\alpha \phi \rangle_0 \\
&\leq C \sum_{i=1}^d \|(\partial_{x_i} \phi \cdot \nabla) \phi\|_{m-1} \|\phi\|_m \\
&\leq C(m, d) \|\nabla \phi\|_{L^\infty} \|\phi\|_m^2.
\end{aligned}$$

□

3 Connection between FBSDEs and the Navier-Stokes Equation

3.1 FBSDEs with coefficients in Sobolev Spaces

Assume that $\nu > 0$ and that

$$b \in C([0, T]; H^m) \cap L^2(0, T; H^{m+1}), \phi \in L^1(0, T; L^2(\mathbb{R}^d)), \psi \in L^2(\mathbb{R}^d), \quad (3.1)$$

for some integer $m > d/2$. Consider the following FBSDE:

$$\begin{cases} dX_s(t, x) = b(s, X_s(t, x)) ds + \sqrt{\nu} dW_s, & s \in [t, T]; \\ X_t(t, x) = x; \\ -dY_s(t, x) = \phi(s, X_s(t, x)) ds - \sqrt{\nu} Z_s(t, x) dW_s, & s \in [t, T]; \\ Y_T(t, x) = \psi(X_T(t, x)). \end{cases} \quad (3.2)$$

Since $H^m \hookrightarrow C^{0,\delta}$ and $H^{m+1} \hookrightarrow C^{1,\delta}$ for $m > d/2$, in view of [27, Theorems 3.4.1 and 4.5.1], the *forward* SDE is well posed for each $(t, x) \in [T_0, T] \times \mathbb{R}^d$, and the unique solution in relevance to the initial data $(t, x) \in [T_0, T] \times \mathbb{R}^d$ defines a stochastic flow of homeomorphisms. Since the function ϕ is only measurable, the following lemma serves to give a clear meaning of the composition $\phi(s, X_s(t, x))$.

Lemma 3.1. *Assume that $m > d/2$ and $b \in C([0, T]; H^m) \cap L^2(0, T; H^{m+1})$. Then there are two positive constants κ and K which only depend on $\|\operatorname{div} b\|_{L^1(0, T; L^\infty)}$, such that for all $t \in [0, T]$, $s \in [t, T]$, $(\varphi, \eta) \in L^1(\mathbb{R}^l) \times L^2([0, T] \times \mathbb{R}^d; \mathbb{R}^l)$, and $l \in \mathbb{Z}^+$, we have*

$$\kappa \|\varphi\|_{L^1(\mathbb{R}^l)} \leq \int_{\mathbb{R}^d} E[|\varphi(X_s(t, x))|] dx \leq K \|\varphi\|_{L^1(\mathbb{R}^l)}, \quad (3.3)$$

$$\kappa \|\eta\|_{L^1([t, T] \times \mathbb{R}^l)} \leq \int_{\mathbb{R}^d} \int_t^T E[|\eta(s, X_s(t, x))|] ds dx \leq K \|\eta\|_{L^1([t, T] \times \mathbb{R}^l)}. \quad (3.4)$$

Our preceding lemma weakens the assumptions on b of [2, Theorem 14.3], where $b(t, \cdot) \equiv b(\cdot)$ is time invariant and is required to lie in the more regular space $C^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$. Since $b(t, x)$ is not necessarily uniformly Lipschitz continuous in x , the stability of X with respect to the coefficient b has to be proved very carefully and the proof of [2] has to be generalized accordingly. we give a probabilistic proof in the appendix.

Remark 3.1. From Lemma 3.1, we see that Lebesgue's measure transported by the flow $\{X_s(t, x), s \in [t, T]\}$ results in a group of measures $\{\mu_s, s \in [t, T]\}$ satisfying for any Borel measurable set $A \subset \mathbb{R}^d$,

$$\mu_s(A) = \int_{\mathbb{R}^d} E[1_A(X_s(t, x))] dx.$$

These measures are all equivalent to Lebesgue measure and the exponential rate of compression or dilation are governed by the divergence of b . This is similar to that of a system of ordinary differential equations (see [13]). On the other hand, thanks to Lemma 3.1, our FBSDE (3.2) makes sense under assumption (3.1), i.e., the *forward* SDE is well posed for each $(t, x) \in [0, T] \times \mathbb{R}^d$ and for each $t \in [0, T]$ the BSDE is well posed for almost every $x \in \mathbb{R}^d$.

3.2 Definition of the solution to system (1.3)

First, let us consider the following trivial BSDE on $[0, \infty]$:

$$-dS_t = g_t dt - dM_t; \quad S_\infty = 0. \quad (3.5)$$

In the conventional sense (see [5, 12, 14, 36, 37]), a solution of BSDE is always defined as a pair of processes (S, M) and the process M serves to guarantee the adaptedness of S . However, since FBSDEs (1.3) only involves \tilde{Y}_0 rather than the whole process \tilde{Y} , we are interested only in determining S_0 instead of the whole process S for such kind of BSDEs on $[0, \infty]$.

Different from [5, 12, 14, 36, 37], we assume that $\{g_t, t \in (0, \infty)\}$ is an \mathbb{R}^d -valued \mathcal{F}_t^B -adapted process such that

$$\int_\varepsilon^\infty E[|g_t|] dt < \infty, \quad \forall \varepsilon > 0 \text{ and } \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty E[g_t] dt < \infty. \quad (3.6)$$

The class \mathcal{E} is defined as the totality of BSDEs (3.5) with the drift satisfying the preceding assumption.

Each BSDE (3.5) lying in \mathcal{E} can be written into the integral form:

$$S_t = \int_t^\infty g_s ds - \widetilde{M}_t,$$

where as we can not solve it in the whole time interval $[0, \infty)$ at once, we solve it with a pair (S, \widetilde{M}) instead of (S, M) , with $E[\widetilde{M}_t | \mathcal{F}_t^B] = 0, \forall t \in (0, \infty)$. Therefore, we have

$$S_t = E\left[\int_t^\infty g_s ds \middle| \mathcal{F}_t^B\right], \quad \widetilde{M}_t = \int_t^\infty g_s ds - S_t, \quad \forall t \in (0, \infty).$$

S_0 is defined as follows

$$S_0 := \lim_{\varepsilon \downarrow 0} E[S_\varepsilon], \quad (3.7)$$

while \widetilde{M}_0 is not necessary for us to know its meaning in this work.

Now, our trivial BSDE (3.5) makes sense and the definition of the solution does not conflict with the common cases (see [5, 12, 14, 36, 37]). Further, we may consider more general nonlinear cases such as $g_s = g(s, S_s, \widetilde{M}_s)$ which may be used to describe more general operators like in Lemma 3.2 below. However, we do not seek such a generality in this paper.

Definition 3.1. We say that $(X, Y, Z, \widetilde{Y}_0)$ is a solution (local solution, respectively) of FBSDS (1.3) if for almost every $(t, x) \in [0, T] \times \mathbb{R}^d$ ($(t, x) \in (T_0, T] \times \mathbb{R}^d$ for some $T_0 \in (0, T)$, respectively), the BSDE on the infinite time interval belongs to class \mathcal{E} , and

$$(X(\cdot(t, x)), Y(\cdot(t, x)), Z(\cdot(t, x))) \in S^2([t, T]; \mathbb{R}^d) \times S^2([t, T]; \mathbb{R}^d) \times L^2_{\mathcal{F}}(t, T; \mathbb{R}^{d \times d})$$

such that the first two stochastic differential equations of (1.3) on $[0, T]$ (any subinterval $[T_1, T]$ with $T_1 \in (T_0, T)$, respectively) hold almost surely.

If both the BSDE on the infinite time interval $[0, \infty]$ and its unknown variable \widetilde{Y} do not appear in FBSDS (1.3), then Definition 3.1 becomes automatically the definition of the solutions (local solutions, respectively) for an FBSDE by abandoning all the requirements on both \widetilde{Y}_0 and the BSDE on the infinite time interval.

To specify the regularity of the solutions, we introduce the following definition.

Definition 3.2. $(X, Y, Z, \widetilde{Y}_0)$ is called a (local, respectively) H^m -solution of FBSDS (1.3) if it is a solution (local solution, respectively) of FBSDS (1.3) (on some time interval $(T_0, T]$, respectively) with $\widetilde{Y}_0 \in L^2(0, T; H^{m-1})$ ($\widetilde{Y}_0 \in L^2_{\text{loc}}(T_0, T; H^{m-1})$, respectively) and for each $t \in [0, T]$ ($t \in (T_0, T]$, respectively) and almost every $x \in \mathbb{R}^d$, $\{Y_s(t, x), s \in [t, T]\} \in L^\infty_{\mathcal{F}}(t, T; \mathbb{R}^d)$.

Remark 3.2. In FBSDS (1.3), the regularity of \widetilde{Y}_0 depends on that of $Y_t(t, x)$. However, Proposition 4.3 below shows that the regularity of \widetilde{Y}_0 dominates that of (Y, Z) . This explains why our H^m -solutions of FBSDS (1.3) only require the regularity of \widetilde{Y}_0 in Definition 3.2. On the other hand, Definition 3.2 requires the uniform boundedness of the unknown process $Y(t, x)$. Indeed, in this work we only pursue the H^m -solution ($m > d/2$, $d = 2$ or 3) of which $Y(t, x)$ is naturally uniformly bounded (see Theorem 3.3 below); moreover, the uniform boundedness is only required for the particular case $m = 2$ (see Proposition 4.3 below). Compared with the standard assumptions for the solvability of FBSDEs (see [1, 1, 20, 29, 35, 43]), it can be viewed as a compensation for the lack of continuity of the function $f(s, x)$ with respect to x .

3.3 Connections between FBSDS and Navier-Stokes equation

Before we show the connections between the FBSDSs and Navier-Stokes equations, we give a probabilistic representation for an integral operator.

Lemma 3.2. For $\phi, \psi \in H^m$ with $m > d/2 + 1$, define

$$\xi(x) := \nabla(-\Delta)^{-1} \operatorname{div} \operatorname{div}(\phi \otimes \psi)(x) = \sum_{i,j=1}^d \nabla(-\Delta)^{-1} \partial_{x^i} \partial_{x^j} (\phi^i(x) \psi^j(x)).$$

Then, the following BSDE :

$$\begin{cases} -d\widetilde{Y}_s(x) = \frac{27}{2s^3} \phi^i \psi^j (x + B_s) \left(B_s^j - B_{\frac{2s}{3}}^j \right) \left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i \right) B_{\frac{s}{3}} ds \\ \quad - dM_s, \quad s \in (0, \infty) \\ \widetilde{Y}_\infty(x) = 0 \end{cases} \quad (3.8)$$

belongs to class \mathcal{E} for each $x \in \mathbb{R}^d$ and there holds

$$\xi(x) = \tilde{Y}_0(x), \quad \forall x \in \mathbb{R}^d. \quad (3.9)$$

Proof. For $m > d/2 + 1$, H^m is a Banach algebra embedded into $H^{2,\gamma}$ for some $\gamma > d$ and also into $C^{1,\delta}(\mathbb{R}^d)$ for some $\delta \in (0, 1)$, so $\phi^i \psi \in H^m \cap H^{m,1}$ which implies $\partial_{x^i} \partial_{x^j} (\phi^i(x) \psi^j(x)) \in H^{0,\gamma}(\mathbb{R}) \cap H^{0,1}(\mathbb{R})$, $i, j = 1, \dots, d$. By the singular integral operator theory (see [39]) and Lemma 2.1, we can check that $\xi \in H^{m-1} \cap C^{0,\delta}(\mathbb{R}^d)$.

For each $\varepsilon > 0$, we have

$$\begin{aligned} & E \left[\int_{\varepsilon}^{\infty} \frac{27}{2s^3} \left| \phi^i \psi^j(x + B_s) \left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i \right) \left(B_s^j - B_{\frac{2s}{3}}^j \right) B_{\frac{s}{3}} \right| ds \right] \\ & \leq C(\|\phi\|_m, \|\psi\|_m) \int_{\varepsilon}^{\infty} \frac{27}{2s^3} E \left[\left| \left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i \right) \left(B_s^j - B_{\frac{2s}{3}}^j \right) B_{\frac{s}{3}} \right| \right] ds \\ & \leq C \int_{\varepsilon}^{\infty} \frac{1}{s^{3/2}} ds < \infty. \end{aligned}$$

Thus, our BSDE (3.8) is well-posed on the interval $[\varepsilon, \infty]$ and we need only prove

$$\xi(x) = \lim_{\varepsilon \downarrow 0} E \left[\tilde{Y}_{\varepsilon}(x) \right]. \quad (3.10)$$

On the other hand, we have

$$\tilde{Y}_{\varepsilon}(x) = E \left[\int_{\varepsilon}^{\infty} \frac{27}{2s^3} \phi^i \psi^j(x + B_s) \left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i \right) \left(B_s^j - B_{\frac{2s}{3}}^j \right) B_{\frac{s}{3}} ds \middle| \mathcal{F}_{\varepsilon}^B \right]. \quad (3.11)$$

Careful calculations yield that, for each $i, j, k = 1, 2, \dots, d$,

$$\begin{aligned} & E \left[\frac{27}{2s^3} \phi^i \psi^j(x + B_s) \left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i \right) \left(B_s^j - B_{\frac{2s}{3}}^j \right) B_{\frac{s}{3}}^k \right] \\ & = \frac{27}{2s^3} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi^i \psi^j(x + y + z + r) y^i z^j r^k (2\pi s/3)^{-3d/2} e^{-\frac{3(|y|^2 + |z|^2 + |r|^2)}{2s}} dy dz dr \\ & = -\frac{9}{2s^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi^i \psi^j(y) z^j r^k (2\pi s/3)^{-3d/2} \partial_{y^i} e^{-\frac{3(|y-x-z-r|^2 + |z|^2 + |r|^2)}{2s}} dy dz dr \\ & = \frac{9}{2s^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{y^i} (\phi^i \psi^j)(y) z^j r^k (2\pi s/3)^{-3d/2} e^{-\frac{3(|y-x-z-r|^2 + |z|^2 + |r|^2)}{2s}} dy dz dr \\ & = -\frac{3}{2s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{y^i} (\phi^i \psi^j)(y) r^k (2\pi s/3)^{-3d/2} e^{-\frac{3|y-x-r-z|^2}{2s}} \partial_{z^j} e^{-\frac{3(|z|^2 + |r|^2)}{2s}} dy dz dr \\ & = -\frac{3}{2s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{y^i} (\phi^i \psi^j)(y) r^k (2\pi s/3)^{-3d/2} \partial_{y^j} e^{-\frac{3|y-x-r-z|^2}{2s}} e^{-\frac{3(|z|^2 + |r|^2)}{2s}} dz dy dr \\ & = \frac{3}{2s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{y^i} \partial_{y^j} (\phi^i \psi^j)(y) r^k (4\pi s/3)^{-d/2} e^{-\frac{3|y-x-r|^2}{4s}} (2\pi s/3)^{-d/2} e^{-\frac{3|r|^2}{2s}} dy dr \\ & = -\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{y^i} \partial_{y^j} (\phi^i \psi^j)(y) (4\pi s/3)^{-d/2} e^{-\frac{3|y-x-r|^2}{4s}} (2\pi s/3)^{-d/2} \partial_{r^k} e^{-\frac{3|r|^2}{2s}} dy dr \\ & = -\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{y^i} \partial_{y^j} (\phi^i \psi^j)(y) (4\pi s/3)^{-d/2} e^{-\frac{3|r|^2}{4s}} (2\pi s/3)^{-d/2} \partial_{y^k} e^{-\frac{3|y-x-r|^2}{2s}} dr dy \\ & = -\frac{1}{2} \int_{\mathbb{R}^d} \partial_{y^i} \partial_{y^j} (\phi^i \psi^j)(y) (2\pi s)^{-d/2} \partial_{y^k} e^{-\frac{|y-x|^2}{2s}} dy \\ & = \frac{1}{2s} \int_{\mathbb{R}^d} \partial_{x^i} \partial_{x^j} (\phi^i \psi^j)(y + x) y^k (2\pi s)^{-d/2} e^{-\frac{|y|^2}{2s}} dy \\ & = \frac{1}{2s} E \left[\partial_{x^i} \partial_{x^j} (\phi^i \psi^j)(x + B_s) B_s^k \right], \quad s > 0. \end{aligned} \quad (3.12)$$

Moreover, we have

$$\begin{aligned}
& s^{-1} \int_{\mathbb{R}^d} |\partial_{x^i} \partial_{x^j} (\phi^i \psi^j)(y+x) y^k (2\pi s)^{-d/2} e^{-\frac{|y|^2}{2s}}| dy \\
& \leq \frac{C}{s^{(d+2)/2}} \|\partial_{x^i} \partial_{x^j} (\phi^i \psi^j)\|_{0,q} \sqrt{s} s^{\frac{d}{2}(1-\frac{1}{q})} \\
& \leq C \|\partial_{x^i} \partial_{x^j} (\phi^i \psi^j)\|_{0,q} s^{-\frac{1}{2}-\frac{d}{2q}}, \quad q \in [1, \gamma]
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
& \int_0^\infty s^{-1} E \left[|\partial_{x^i} \partial_{x^j} (\phi^i \psi^j)(x+B_s) B_s^k| \right] ds \\
& = \int_0^\infty s^{-1} \int_{\mathbb{R}^d} |\partial_{x^i} \partial_{x^j} (\phi^i \psi^j)(y+x) y^k (2\pi s)^{-d/2} e^{-\frac{|y|^2}{2s}}| dy ds \\
& = C \int_0^1 \|\partial_{x^i} \partial_{x^j} (\phi^i \psi^j)\|_{0,\gamma} s^{-\frac{1}{2}-\frac{d}{2\gamma}} ds + C \int_1^\infty \|\partial_{x^i} \partial_{x^j} (\phi^i \psi^j)\|_{0,1} s^{-\frac{d+1}{2}} ds \\
& \leq C \left(\|\partial_{x^i} \partial_{x^j} (\phi^i \psi^j)\|_{m-2} + \|\partial_{x^i} \partial_{x^j} (\phi^i \psi^j)\|_{m-2,1} \right).
\end{aligned} \tag{3.14}$$

Therefore,

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} E \left[\tilde{Y}_\varepsilon^k(x) \right] &= \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty \frac{1}{2s} \int_{\mathbb{R}^d} \partial_{x^i} \partial_{x^j} (\phi^i \psi^j)(y+x) y^k (2\pi s)^{-d/2} e^{-\frac{|y|^2}{2s}} dy ds \\
&= \int_0^\infty \frac{1}{2s} \int_{\mathbb{R}^d} \partial_{x^i} \partial_{x^j} (\phi^i \psi^j)(y+x) y^k (2\pi s)^{-d/2} e^{-\frac{|y|^2}{2s}} dy ds \\
&\quad (\text{by (3.14) and Fubini Theorem}) \\
&= \int_{\mathbb{R}^d} \partial_{x^i} \partial_{x^j} (\phi^i \psi^j)(x+y) y^k \int_0^\infty \frac{1}{2s} (2\pi s)^{-d/2} e^{-\frac{|y|^2}{2s}} ds dy \\
&= C_d \int_{\mathbb{R}^d} \partial_{x^i} \partial_{x^j} (\phi^i \psi^j)(x+y) \frac{y^k}{|y|^d} dy \\
&= \partial_{x^k} (-\Delta)^{-1} \partial_{x^i} \partial_{x^j} (\phi^i \psi^j)(x)
\end{aligned} \tag{3.15}$$

which coincides with the convolution representation of the operator $\nabla(-\Delta)^{-1}$ described in [30, Page 31]. Hence, BSDE (3.8) belongs to class \mathcal{E} and by (3.14), $\tilde{Y}_0 \in C(\mathbb{R}^d)$ on account of the continuity of the translation operator on $L^p(\mathbb{R}^d)$, $p \in [1, \infty)$. Moreover, we have

$$\xi(x) = \nabla(-\Delta)^{-1} \partial_{x^i} \partial_{x^j} (\phi^i(x) \psi^j(x)) = \lim_{\varepsilon \downarrow 0} E \left[\tilde{Y}_\varepsilon(x) \right], \quad \forall x \in \mathbb{R}^d,$$

which completes the proof. \square

Remark 3.3. We have $\tilde{Y}_0 \in C(\mathbb{R}^d)$ from (3.14) and the continuity of the translation operator on $L^p(\mathbb{R}^d)$ for $p \in [1, \infty)$. However, in this work we would rather use more properties of \tilde{Y}_0 basing on harmonic analysis. In fact, the operator $\mathbf{P} := \mathbf{I} - \nabla \Delta^{-1} \text{div}$ is the well-known Leray-Hodge projection, where \mathbf{I} is the identity operator. Note that the singular integral operator \mathbf{P} (see [30, 39]) is a bounded transformation in $H^{n,q}$ for $q \in (1, \infty)$ and $n \in \mathbb{Z}$. Define $\mathbf{P}^\perp := \mathbf{I} - \mathbf{P}$. We have in Lemma 3.2 that $\xi = -\mathbf{P}^\perp(\text{div}(\phi \otimes \psi))$.

Remark 3.4. For the defined ξ , there exists a scalar-valued function $\eta \in H^m(\mathbb{R}^d; \mathbb{R})$ such that $\xi = \nabla \eta$. Indeed, we may take $\eta(x) =: (-\Delta)^{-1} \partial_{x^i} \partial_{x^j} (\phi^i \psi^j)(x)$ which, by the second order Elliptic partial differential equation theory (see [19]), lies in $H^m(\mathbb{R}^d; \mathbb{R})$.

Define the following operator θ : for any map g defined on $[0, \infty) \times [0, \infty) \times \mathbb{R}^d$,

$$\theta_g(t, x) := g(t, t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d.$$

Our main result is stated as follows.

Theorem 3.3. Let $\nu > 0$, $G \in H_\sigma^m$, and $f \in L^2(0, T; H_\sigma^{m-1})$ with $m > d/2$. Then our FBSDE (1.3) admits one and only one local H^m -solution (X, Y, Z, \tilde{Y}_0) on some time interval $(T_0, T]$ with $\theta_Y \in C_{loc}((T_0, T]; H_\sigma^m) \cap L_{loc}^2(T_0, T; H_\sigma^{m+1})$ and $\theta_Z \in C_{loc}((T_0, T]; H_\sigma^{m-1}) \cap L_{loc}^2(T_0, T; H_\sigma^m)$, where T_0 depends on $\nu, T, \|G\|_m$ and $\|f\|_{L^2(0, T; H^{m-1})}$. Moreover, there hold the following representations

$$\theta_Z(t, \cdot) = \nabla \theta_Y(t, \cdot), Y_s(t, \cdot) = \theta_Y(s, X_s(t, \cdot)) \text{ and } Z_s(t, \cdot) := \theta_Z(s, X_s(t, \cdot)), \quad T_0 < t \leq s \leq T,$$

and there exists some scalar-valued function \tilde{p} such that $\nabla \tilde{p} = \tilde{Y}_0$ and $(\theta_Y, \theta_Z, \tilde{p})$ satisfies

$$\begin{aligned} \theta_Y(r, X_r(t, x)) &= G(X_T(t, x)) + \int_r^T [f(s, X_s(t, x)) + \nabla \tilde{p}(s, X_s(t, x))] ds \\ &\quad - \sqrt{\nu} \int_r^T \theta_Z(s, X_s(t, x)) dW_s, \quad T_0 < t \leq r \leq T, \text{ a.e. } x \in \mathbb{R}^d, \text{ a.s..} \end{aligned}$$

In addition, (θ_Y, \tilde{p}) coincides with the unique strong solution to Navier-Stokes equation:

$$\begin{cases} \partial_t \theta_Y + \frac{\nu}{2} \Delta \theta_Y + (\theta_Y \cdot \nabla) \theta_Y + \nabla \tilde{p} + f = 0, & T_0 < t \leq T; \\ \nabla \cdot \theta_Y = 0, & \theta_Y(T) = G. \end{cases} \quad (3.16)$$

Remark 3.5. In Theorem 3.3, we only have the connections for the strong solutions. For the case of $m \leq d/2$, we could not show that $\theta_Y(t, \cdot)$ takes values in $W_{loc}^{1, \infty}$, and thus we do not know whether the forward SDE of FBSDE (1.3) is well-posed or whether the assertions of Lemma 3.1 are still true.

4 Existence and uniqueness

4.1 Auxiliary results

Consider the following coupled FBSDE:

$$\begin{cases} dX_s(t, x) = [b(s, X_s(t, x)) + \alpha Y_s(t, x)] ds + \sqrt{\nu} dW_s, & s \in [t, T]; \\ X_t(t, x) = x; \\ -dY_s(t, x) = \phi(s, X_s(t, x)) ds - \sqrt{\nu} Z_s(t, x) dW_s, & s \in [t, T]; \\ Y_T(t, x) = \psi(X_T(t, x)), \end{cases} \quad (4.1)$$

where $\nu > 0$ and α are constants.

We define a bounded and a locally bounded solutions to FBSDE (4.1).

Definition 4.1. We say (X, Y, Z) is a bounded solution (locally bounded solution, respectively) of FBSDE (4.1) if for each $t \in [0, T]$ ($t \in (T_0, T]$ for some $T_0 \in (0, T)$, respectively) and almost every $x \in \mathbb{R}^d$,

$$(X(\cdot, t, x), Y(\cdot, t, x), Z(\cdot, t, x)) \in S^2([t, T]; \mathbb{R}^d) \times S^2([t, T]; \mathbb{R}^d) \times L_{\mathcal{F}}^2(t, T; \mathbb{R}^{d \times d})$$

such that the forward SDE and BSDE on $[0, T]$ (any $[T_1, T] \subset (T_0, T)$, respectively) hold almost surely.

Lemma 4.1. Let $b, \phi \in C_c^\infty(\mathbb{R}^{d+1})$ and $\psi \in C_c^\infty(\mathbb{R}^d)$. Then for each integer $m > d/2$, FBSDE (4.1) admits a unique locally bounded solution (X, Y, Z) on some time interval $(\tau, T]$ with $T - \tau$ ($\tau \in [0, T)$) continuously depending on $\|\phi\|_{L^2(0, T; H^{m-1})}$, $\|b\|_{C([0, T]; H^m)}$, $\|\psi\|_m, \nu$ and α . Moreover, we have

$$\theta_Y \in C_{loc}((\tau, T]; H^m) \cap L_{loc}^2(\tau, T; H^{m+1}), \quad (4.2)$$

and for each $t \in (\tau, T]$, almost all $x \in \mathbb{R}^d$ and all $r \in [t, T]$

$$\theta_Y(r, X_r(t, x)) = \psi(X_T(t, x)) + \int_r^T \phi(s, X_s(t, x)) ds - \sqrt{\nu} \int_r^T \theta_Z(s, X_s(t, x)) dW_s, \text{ a.s.} \quad (4.3)$$

$$\theta_Z(t, x) = \nabla \theta_Y(t, x), Y_r(t, x) = \theta_Y(r, X_r(t, x)), Z_r(t, x) = \theta_Z(r, X_r(t, x)), \text{ a.s..} \quad (4.4)$$

In particular, if $\alpha = 0$, we are allowed to take $T_0 = 0$ and we have

$$\theta_Y \in C([0, T]; H^m) \cap L^2(0, T; H^{m+1}) \cap C^\infty([0, T] \times \mathbb{R}^d).$$

Lemma 4.1 might exist elsewhere, but we have not found it. For the reader's convenience, a proof is sketched in the appendix.

Remark 4.1. In Lemma 4.1, for each $T_0 \in (\tau, T)$ and each $T' \in (T_0, T]$, consider the following FBSDS:

$$\begin{cases} dX_s(T_0, x) = [b(s, X_s(T_0, x)) + \alpha\theta_Y(s, X_s(T_0, x))] ds + \sqrt{\nu} dW_s, & s \in [T_0, T']; \\ X_{T_0}(T_0, x) = x; \\ -dY_s(T_0, x) = \phi(s, X_s(T_0, x)) ds - \sqrt{\nu} Z_s(T_0, x) dW_s, & s \in [T_0, T']; \\ Y_{T'}(T_0, x) = \theta_Y(T', X_{T'}(T_0, x)). \end{cases}$$

It has the same unique locally bounded solution $(X.(T_0, x), Y.(T_0, x), Z.(T_0, x))$ as FBSDS (4.1). Define the following new equivalent probability measure:

$$d\mathbb{Q}^{T_0, x} := \exp \left(-\nu^{-\frac{1}{2}} \int_{T_0}^T (b + \alpha\theta_Y)(s, X_s(T_0, x)) dW_s - \frac{1}{2} \nu^{-1} \int_{T_0}^T |(b + \alpha\theta_Y)(s, X_s(T_0, x))|^2 ds \right) d\mathbb{P}.$$

In view of Girsanov Theorem, the process

$$\widetilde{W}_r := \nu^{-\frac{1}{2}} \int_{T_0}^r (b + \alpha\theta_Y)(s, X_s(T_0, x)) ds + W_r, \quad r \in [T_0, T]$$

is a standard Brownian motion under $\mathbb{Q}^{T_0, x}$ and we have

$$\begin{cases} dX_s(T_0, x) = \sqrt{\nu} d\widetilde{W}_s, & s \in [T_0, T']; \\ X_{T_0}(T_0, x) = x; \\ -dY_s(T_0, x) = [\phi(s, X_s(T_0, x)) + \theta_Z(b + \alpha\theta_Y)(s, X_s(T_0, x))] ds - \sqrt{\nu} Z_s(T_0, x) d\widetilde{W}_s, & s \in [T_0, T']; \\ Y_{T'}(T_0, x) = u(T', X_{T'}(T_0, x)), \end{cases}$$

where we have used the expression for Z in (4.4). For the unique solution $(X'(T_0, x), Y'(T_0, x), Z'(T_0, x))$ to the following FBSDS:

$$\begin{cases} dX'_s(T_0, x) = \sqrt{\nu} d\widetilde{W}_s, & s \in [T_0, T']; \\ X'_{T_0}(T_0, x) = x; \\ -dY'_s(T_0, x) = I_m[\phi + \theta_Z(b + \alpha\theta_Y)](s, X'_s(T_0, x)) ds - \sqrt{\nu} Z'_s(T_0, x) d\widetilde{W}_s, & s \in [T_0, T']; \\ Y'_{T'}(T_0, x) = I_m u(T', X'_{T'}(T_0, x)), \end{cases}$$

it is not difficult for us to show that for almost every $x \in \mathbb{R}^d$ and all $s \in [T_0, T']$

$$Y'_s(T_0, x) = I_m \theta_Y(s, X'_s(T_0, x)) \text{ and } Z'_s(T_0, x) = I_m \theta_Z(s, X'_s(T_0, x)), \text{ a.s.,}$$

and from Itô's formula, we have

$$\begin{aligned} & E_{\mathbb{Q}^{T_0, x}} \left[|Y'_{T_0}(T_0, x)|^2 + \nu \int_{T_0}^{T'} |Z'_s(T_0, x)|^2 ds \right] \\ &= 2 \int_{T_0}^{T'} E_{\mathbb{Q}^{T_0, x}} [< Y'_s(T_0, x), I_m \phi(s, X'_s(T_0, x)) + I_m \{ \theta_Z(b + \alpha\theta_Y)(s, X'_s(T_0, x)) \} >] ds \\ &\quad + E_{\mathbb{Q}^{T_0, x}} [|I_m \theta_Y(T', X'_{T'}(T_0, x))|^2], \text{ a.e. } x \in \mathbb{R}^d. \end{aligned}$$

Finally, integrating with respect to x on both sides of the last equality, we obtain the energy equality:

$$\begin{aligned} & \|\theta_Y(T_0)\|_m^2 + \nu \int_{T_0}^{T'} \|\theta_Z(s)\|_m^2 ds \\ &= \|\theta_Y(T')\|_m^2 + 2 \int_{T_0}^{T'} \langle \phi(s) + \theta_Z(b + \alpha\theta_Y)(s), \theta_Y(s) \rangle_{m-1, m+1} ds. \end{aligned}$$

Note that

$$\|\theta_Y(t)\|_m^2 = \|\theta_Z(t)\|_{m-1}^2 + \|\theta_Y(t)\|_0^2 \leq \|\theta_Z(t)\|_{m-1}^2 + \|\theta_Y\|_{m-1}^2.$$

Remark 4.2. From Lemma 4.1, we see θ_Y and θ_Z are all deterministic functions. Therefore, for each semimartingale $\{X'_s(t, x), s \in [t, T]\}$ of the form

$$X'_s(t, x) = x + \int_t^s \varphi_r(t, x) dr + \int_t^s \sqrt{\nu} dW_r, \quad T_0 < t \leq s \leq T$$

with $\{\varphi_s(t, x), s \in [t, T]\}$ being bounded and predictable, it is interesting to understand $(\theta_Y, \theta_Z)(s, X'_s(t, x))$ in the FBSDE framework. Indeed, Analogous to the preceding remark, define the following equivalent probability measure:

$$\begin{aligned} d\mathbb{Q}^{t,x} := \exp & \left(\frac{1}{\sqrt{\nu}} \int_t^T [(b + \alpha\theta_Y)(s, X'_s(t, x)) - \varphi_s(t, x)] dW_s \right. \\ & \left. - \frac{1}{2\nu} \int_t^T |(b + \alpha\theta_Y)(s, X'_s(t, x)) - \varphi_s(t, x)|^2 ds \right) d\mathbb{P}. \end{aligned}$$

Then in view of Girsanov theorem, there is a standard Brownian motion $(W', \mathbb{Q}^{t,x})$ such that

$$X'_s(t, x) = x + \int_t^s (b + \alpha\theta_Y)(r, X'_r(t, x)) dr + \int_t^s \sqrt{\nu} dW'_r, \quad T_0 < t \leq s \leq T.$$

From Lemma 4.1, we have

$$\begin{aligned} \theta_Y(\tau, X'_\tau(t, x)) &= \psi(X'_T(t, x)) + \int_\tau^T \phi(s, X'_s(t, x)) ds - \sqrt{\nu} \int_\tau^T \theta_Z(s, X'_s(t, x)) dW'_s \\ &= \psi(X'_T(t, x)) + \int_\tau^T \left\{ [\phi(s, X'_s(t, x)) + \theta_Z(b + \alpha\theta_Y)](s, X'_s(t, x)) - Z\varphi_s(t, x) \right\} ds \\ &\quad - \sqrt{\nu} \int_\tau^T \theta_Z(s, X'_s(t, x)) dW_s, \quad t \leq \tau \leq T. \end{aligned}$$

Lemma 4.2. Let $b \in C([T_0, T]; H^m) \cap L^2(T_0, T; H^{m+1})$, $\phi \in L^2(T_0, T; H^{m-1})$, and $\psi \in H^m$, with $m > d/2$. Then, for each $t \in [T_0, T)$ and almost every $x \in \mathbb{R}^d$, the following FBSDE:

$$\begin{cases} dX_s(t, x) = b(s, X_s(t, x)) ds + \sqrt{\nu} dW_s, & T - \varepsilon \leq t \leq s \leq T; \\ X_t(t, x) = x; \\ -dY_s(t, x) = \phi(s, X_s(t, x)) ds - \sqrt{\nu} Z_s(t, x) dW_s, & s \in [t, T]; \\ Y_T(t, x) = \psi(X_T(t, x)) \end{cases} \quad (4.5)$$

admits one and only one solution

$$(X(t, x), Y(t, x), Z(t, x)) \in S^2(T_0, T; \mathbb{R}^d) \times S^2(T_0, T; \mathbb{R}^d) \times L^2_{\mathcal{F}}(T_0, T; \mathbb{R}^d),$$

and for this solution (X, Y, Z) , there hold (4.2), (4.3) and (4.4) with τ being replaced with T_0 therein.

Proof. For $m > d/2$, $H^m \hookrightarrow C^{0,\delta}$, $H^{m+1} \hookrightarrow C^{1,\delta}$. By Theorems 3.4.1 and 4.5.1 of [27], the forward SDE is well posed for each $(t, x) \in [T_0, T] \times \mathbb{R}^d$ and defines a stochastic flow of homeomorphisms. Moreover, from Lemma 3.1 and Remark 3.1, the backward SDE is also well posed for every $x \in \mathbb{R}^d/F_t$ with Lebesgue's measure of F_t being zero. Therefore, for each $(t, x) \in [T_0, T] \times (\mathbb{R}^d/F_t)$, FBSDE (4.5) has unique solution

$$(X(t, x), Y(t, x), Z(t, x)) \in S^2(T_0, T; \mathbb{R}^d) \times S^2(T_0, T; \mathbb{R}^d) \times L^2_{\mathcal{F}}(T_0, T; \mathbb{R}^d).$$

For each $(t, x) \in [T_0, T) \times (\mathbb{R}^d/F_t)$, define the following equivalent probability measure:

$$d\mathbb{Q}^{t,x} := \exp \left(-\frac{1}{\sqrt{\nu}} \int_t^T b(s, X_s(t, x)) dW_s - \frac{1}{2\nu} \int_t^T |b(s, X_s(t, x))|^2 ds \right) d\mathbb{P}.$$

Then there is a standard brownian motion $(W', \mathbb{Q}^{t,x})$ such that FBSDE (4.5) is written into the following form:

$$\begin{cases} dX_s(t, x) = \sqrt{\nu} dW'_s, & T - \varepsilon \leq t \leq s \leq T; \\ X_t(t, x) = x; \\ -dY_s(t, x) = \phi(s, X_s(t, x)) + Z_s(t, x)b(s, X_s(t, x)) ds - \sqrt{\nu} Z_s(t, x) dW'_s, & s \in [t, T]; \\ Y_T(t, x) = \psi(X_T(t, x)). \end{cases} \quad (4.6)$$

Choose a sequence $\{(b^n, \phi^n, \psi^n), n \in \mathbb{Z}^1\} \subset C_c^\infty(\mathbb{R}^{d+1}) \times C_c^\infty(\mathbb{R}^{d+1}) \times C_c^\infty(\mathbb{R}^d)$ satisfying

$$\lim_{n \rightarrow \infty} (\|b^n - b\|_{C([t, T]; H^m)} + \|b^n - b\|_{L^2([t, T]; H^{m+1})} + \|\phi^n - \phi\|_{L^2(t, T; H^{m-1})} + \|\psi^n - \psi\|_{H^m}) = 0.$$

Let (X, Y^n, Z^n) be the unique solution of FBSDE (4.6) with (b, ϕ, ψ) being replaced with (b^n, ϕ^n, ψ^n) . Then for each n , we have

$$\theta_{Y^n} \in C([t, T]; H^m) \cap L^2(t, T; H^{m+1}),$$

and for each $x \in \mathbb{R}^d$ and all $t \leq r \leq T$,

$$\begin{aligned} \theta_{Y^n}(r, X_r(t, x)) &= \psi^n(X_T(t, x)) + \int_r^T (\phi^n + \theta_{Z^n} b^n)(s, X_s(t, x)) ds - \sqrt{\nu} \int_r^T \theta_{Z^n}(s, X_s(t, x)) dW'_s, \\ \theta_{Z^n}(t, x) &= \nabla \theta_{Y^n}(t, x), Y_r^n(t, x) = \theta_{Y^n}(r, X_r(t, x)), Z_r^n(t, x) = \theta_{Z^n}(r, X_r(t, x)). \end{aligned}$$

On the one hand, using Itô's formula, we have, for each $(s, x) \in [t, T] \times \mathbb{R}^d / F_t$

$$\begin{aligned} &|Y_s^n(t, x) - Y_s(t, x)|^2 + \nu \int_s^T |Z_r^n(t, x) - Z_r(t, x)|^2 dr \\ &= |\psi(X_T(t, x)) - \psi^n(X_T(t, x))|^2 - \int_s^T 2\nu \langle Y_r^n(t, x) - Y_r(t, x), Z_r^n(t, x) - Z_r(t, x) \rangle dW'_r \\ &\quad + \int_s^T 2 \langle Y_r^n(t, x) - Y_r(t, x), (\phi^n - \phi)(r, X_r(t, x)) + Z_r^n(t, x)b^n(r, X_r(t, x)) \\ &\quad \quad - Z_r(t, x)b(r, X_r(t, x)) \rangle dr, \end{aligned} \quad (4.7)$$

and therefore, using the BDG inequality,

$$\begin{aligned} &E_{\mathbb{Q}^{t,x}} \left[\sup_{\tau \in [s, T]} |Y_\tau^n(t, x) - Y_\tau(t, x)|^2 + \nu \int_s^T |Z_r^n(t, x) - Z_r(t, x)|^2 dr \right] \\ &\leq E_{\mathbb{Q}^{t,x}} \left[|\psi(X_T(t, x)) - \psi^n(X_T(t, x))|^2 \right. \\ &\quad + \frac{1}{2} \sup_{\tau \in [s, T]} |Y_\tau^n(t, x) - Y_\tau(t, x)|^2 + C \int_s^T |Z_r^n(t, x) - Z_r(t, x)|^2 dr \\ &\quad + C \int_s^T |Y_r^n(t, x) - Y_r(t, x)| (|\phi^n - \phi|(r, X_r(t, x)) + \|b^n - b\|_{C([t, T]; H^m)} |Z_r(t, x)| \\ &\quad \quad \left. + |Z_r^n(t, x) - Z_r(t, x)|) dr \right], \end{aligned} \quad (4.8)$$

with the constants C s being independent of n . Combining (4.7) and (4.8), we have

$$\begin{aligned} &E_{\mathbb{Q}^{t,x}} \left[\sup_{\tau \in [t, T]} |Y_\tau^n(t, x) - Y_\tau(t, x)|^2 + \nu \int_t^T |Z_r^n(t, x) - Z_r(t, x)|^2 dr \right] \\ &\leq CE_{\mathbb{Q}^{t,x}} \left[\int_t^T (|\phi^n(r, X_r(t, x)) - \phi(r, X_r(t, x))|^2 + \|b^n - b\|_{C([t, T]; H^m)}^2 |Z_r(t, x)|^2) dr \right] \\ &\quad + C \|\psi - \psi^n\|_{H^m}^2 \longrightarrow 0, \text{ as } n \rightarrow \infty, \quad x \in \mathbb{R}^d / F_t. \end{aligned} \quad (4.9)$$

On the other hand, setting

$$(Y^{nk}, Z^{nk}, b_{nk}, \phi_{nk}, \psi_{nk}) := (Y^n - Y^k, Z^n - Z^k, b_n - b_k, \phi^n - \phi^k, \psi^n - \psi^k).$$

For $n, k \in \mathbb{Z}^+$, we have from Remarks 2.1 and 4.1 that

$$\begin{aligned} & \|\theta_{Y^{nk}}(s)\|_m^2 + \nu \int_s^T \|\theta_{Z^{nk}}(r)\|_m^2 dr \\ &= \|\psi_{nk}\|_m^2 + \int_s^T 2\langle \phi_{nk}(r) + \theta_{Z^n} b_{nk}(r) + \theta_{Z^{nk}} b_k(r), \theta_{Y^{nk}}(r) \rangle_{m-1, m+1} dr \\ &\leq \|\psi_{nk}\|_m^2 + \frac{\nu}{2} \int_s^T (\|\theta_{Z^{nk}}(r)\|_m^2 + \|\theta_{Y^{nk}}(r)\|_m^2) dr \\ &\quad + C(\nu) \left(\int_s^T \|b_{nk}(r)\|_m^2 \|\theta_{Y^n}(r)\|_m^2 dr + \int_s^T \|b_k(r)\|_m^2 \|\theta_{Y^{nk}}(r)\|_m^2 dr + \int_s^T \|\phi_{nk}(r)\|_{m-1}^2 dr \right) \\ &\leq C \left\{ \|\psi_{nk}\|_m^2 + \frac{\nu}{2} \int_s^T (\|\theta_{Z^{nk}}(r)\|_m^2 + \|\theta_{Y^{nk}}(r)\|_m^2) dr \right. \\ &\quad \left. + \int_0^T (\|b_{nk}(r)\|_m^2 + \|\phi_{nk}(r)\|_{m-1}^2) dr + \int_s^T \|\theta_{Y^{nk}}(r)\|_m^2 dr \right\}, \end{aligned}$$

where we have used the the following priori estimate by taking $(b^k, \phi^k, \psi^k) = 0$ in the above,

$$\|\theta_{Y^n}\|_{C([T_0, T]; H^m)} + \|\theta_{Y^n}\|_{L^2(T_0, T; H^{m+1})} \leq C,$$

with the constant C being independent of n . Thus,

$$\begin{aligned} & \sup_{s \in [T_0, T]} \|\theta_{Y^{nk}}(s)\|_m^2 + \nu \int_{T_0}^T \|\theta_{Z^{nk}}(r)\|_m^2 dr \\ &\leq C \left(\|\psi_{nk}\|_m^2 + \int_{T_0}^T (\|b_{nk}(r)\|_m^2 + \|\phi_{nk}(r)\|_{m-1}^2) dr \right) \longrightarrow 0 \text{ as } n, k \rightarrow \infty. \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10), we have

$$\lim_{k \rightarrow \infty} \left(\|\theta_{Y^k} - \theta_Y\|_{C([T_0, T]; H^m)} + \|\theta_{Z^k} - \theta_Z\|_{L^2_{\mathcal{F}}(T_0, T; H^m)} \right) = 0,$$

with $\theta_Z = \nabla \theta_Y$, and furthermore, by taking limits, we can deduce that (4.2), (4.3) and (4.4) hold. We complete the proof. \square

Proposition 4.3. *Assume that $\psi \in H^m$, $b \in C([0, T]; H^m) \cap L^2(0, T; H^{m+1})$, and $\phi \in L^2(0, T; H^{m-1})$ with $m > d/2$. Then FBSDE (4.1) admits a unique locally bounded solution in some time interval $(T_0, T]$ with T_0 depending on $\|\psi\|_m, \|b\|_{C([0, T]; H^m)}, \|\phi\|_{L^2(0, T; H^{m-1})}, \alpha, \nu$ and T . If $\alpha = 0$, $T_0 = 0$. Moreover,*

$$\theta_Y \in C_{loc}((T_0, T]; H^m) \cap L^2_{loc}(T_0, T; H^{m+1}) \text{ and } \theta_Z \in C_{loc}((T_0, T]; H^{m-1}) \cap L^2_{loc}(T_0, T; H^m)$$

satisfy (4.3) and for any $t \in (T_0, T]$, there hold the following energy equality:

$$\begin{aligned} & \|\theta_Y(t)\|_m^2 + \nu \int_t^T \|\theta_Z(s)\|_m^2 ds \\ &= \|\psi\|_m^2 + 2 \int_t^T \langle \theta_Z(b + \alpha \theta_Y)(s), \theta_Y(s) \rangle_{m-1, m+1} ds + 2 \int_t^T \langle \phi(s), \theta_Y(s) \rangle_{m-1, m+1} ds \end{aligned} \tag{4.11}$$

and for almost every $x \in \mathbb{R}^d$ and all $s \in [t, T]$,

$$\theta_Z(t, x) = \nabla \theta_Y(t, x), Y_s(t, x) = \theta_Y(s, X_s(t, x)), Z_s(t, x) = \theta_Z(s, X_s(t, x)), \text{ a.s..} \tag{4.12}$$

In addition, if $m > d/2 + 1$, the locally bounded solution on the time interval $(T_0, T]$ is the unique local solution as well.

Proof. By Lemma 4.2, it remains for us to consider the case $\alpha \neq 0$.

Step 1. We shall prove the existence of the solution. Choose a sequence $(b_n, \phi_n, \psi_n) \in C_c^\infty(\mathbb{R}^{1+d}) \times C_c^\infty(\mathbb{R}^{1+d}) \times C_c^\infty(\mathbb{R}^d)$ such that

$$\lim_{n \rightarrow \infty} \{ \|b_n - b\|_{C([0,T];H^m)} + \|b_n - b\|_{L^2(0,T;H^{m+1})} + \|\phi_n - \phi\|_{L^2(0,T;H^{m-1})} + \|\psi_n - \psi\|_m \} = 0,$$

$$\|b_n\|_{C([0,T];H^m)} \leq C\|b\|_{C([0,T];H^m)}, \quad \|\phi_n\|_{L^2(0,T;H^{m-1})} \leq C\|\phi\|_{L^2(0,T;H^{m-1})}, \quad \|\psi_n\|_m \leq C\|\psi\|_m,$$

and $\|b_n\|_{L^2(0,T;H^{m+1})} \leq C\|b\|_{L^2(0,T;H^{m+1})}$, where C is a universal constant being independent of n . By Lemma 4.1, for each n , FBSDE (4.1) with (b, ϕ, ψ) being replaced with (b_n, ϕ_n, ψ_n) admits a unique local solution (X^n, Y^n, Z^n) on some time interval $(\tau, T]$ such that $(\theta_{Y^n}, \theta_{Z^n})$ satisfies (4.3) associated with (b_n, ϕ_n, ψ_n) . As $T - \tau$ continuously depends on $\|\phi_n\|_{L^2(0,T;H^{m-1})}$, $\|\psi_n\|_m$, $\|b_n\|_{C([0,T];H^m)}$, ν and α , we can choose a uniform τ for all $n \in \mathbb{Z}$. Moreover, we have

$$\theta_{Z^n}(t, x) = \nabla \theta_{Y^n}(t, x), \quad Y_s^n(t, x) = \theta_{Y^n}(s, X_s^n(t, x)) \quad \text{and} \quad Z_s^n(t, x) = \theta_{Z^n}(s, X_s^n(t, x)),$$

and by Remark 4.1,

$$\begin{aligned} & \|\theta_{Y^n}(s)\|_m^2 + \nu \int_s^T \|\theta_{Z^n}(r)\|_m^2 dr \\ &= \|\psi_n\|_m^2 + \int_s^T 2\langle \theta_{Z^n}(b_n + \alpha \theta_{Y^n})(r), \theta_{Y^n}(r) \rangle_{m-1, m+1} dr + \int_s^T 2\langle \phi_n(r), \theta_{Y^n}(r) \rangle_{m-1, m+1} dr \\ &\leq \|\psi_n\|_m^2 + C \int_s^T (\|(b_n + \alpha \theta_{Y^n})(r)\|_m \|\theta_{Z^n}(r)\|_{m-1} \|\theta_{Y^n}(r)\|_{m+1} + \|\phi_n(r)\|_{m-1} \|\theta_{Y^n}(r)\|_{m+1}) dr \\ &\leq C(\nu) \left\{ \int_s^T \left(\|b(r)\|_{L^\infty(0,T;H^m)}^2 + 1 \right) \|\theta_{Y^n}(r)\|_m^2 dr + \alpha^2 \int_s^T \|\theta_{Y^n}(r)\|_m^4 dr \right. \\ &\quad \left. + \int_s^T \|\phi(r)\|_{m-1}^2 dr \right\} + \frac{\nu}{2} \int_s^T \|\theta_{Z^n}(r)\|_m^2 dr + C\|\psi\|_m^2. \end{aligned}$$

Therefore,

$$\begin{aligned} & \|\theta_{Y^n}(s)\|_m^2 + \nu \int_s^T \|\theta_{Z^n}(r)\|_m^2 dr \\ &\leq C(\nu, \phi, \psi) + C(\nu, b) \left(\int_s^T \|\theta_{Y^n}(r)\|_m^2 dr + \alpha^2 \int_s^T \|\theta_{Y^n}(r)\|_m^4 dr \right) \\ &\leq C(\phi, \psi, \nu, b, \alpha, T) + C(\nu, b) \alpha^2 \int_s^T \|\theta_{Y^n}(r)\|_m^4 dr. \end{aligned}$$

In conclusion, there is a constant $T_0 = T_0(\phi, \psi, \nu, b, \alpha, T) \in [\tau, T)$ such that for $C_0 := C(\phi, \psi, \nu, b, \alpha, T)$,

$$\sup_{r \in [s, T]} \|\theta_{Y^n}(r)\|_m^2 + \nu \int_s^T \|\theta_{Z^n}(r)\|_m^2 dr \leq \frac{C_0}{1 - \alpha^2 C_0 C(\nu, b)(T - s)}, \quad \forall s \in (T_0, T]. \quad (4.13)$$

Set

$$(X^{nk}, Y^{nk}, Z^{nk}, b_{nk}, \phi_{nk}, \psi_{nk}) := (X^n - X^k, Y^n - Y^k, Z^n - Z^k, b_n - b_k, \phi_n - \phi_k, \psi_n - \psi_k).$$

Then for each fixed $\varepsilon \in (0, T - T_0)$, we have, for any $s \in (T_0 + \varepsilon, T]$

$$\begin{aligned}
& \|\theta_{Y^{nk}}(s)\|_m^2 + \nu \int_s^T \|\theta_{Z^{nk}}(r)\|_m^2 dr \\
&= \|\psi_{nk}\|_m^2 + \int_s^T 2\langle \theta_{Z^n} b_{nk}(r) + \theta_{Z^{nk}} b_k(r), \theta_{Y^{nk}}(r) \rangle_{m-1, m+1} dr \\
&\quad + \int_s^T 2\alpha \langle \theta_{Z^n} \theta_{Y^{nk}}(r) + \theta_{Z^{nk}} \theta_{Y^k}(r), \theta_{Y^{nk}}(r) \rangle_{m-1, m+1} dr + \int_s^T 2\langle \phi_{nk}(r), \theta_{Y^{nk}}(r) \rangle_{m-1, m+1} dr \\
&\leq \|\psi_{nk}\|_m^2 + \frac{\nu}{2} \int_s^T (\|\theta_{Z^{nk}}(r)\|_m^2 + \|\theta_{Y^{nk}}(r)\|_m^2) dr \\
&\quad + C(\nu) \left\{ \int_s^T \|b_{nk}(r)\|_m^2 \|\theta_{Y^n}(r)\|_m^2 dr + \int_s^T \|b_k(r)\|_m^2 \|\theta_{Y^{nk}}(r)\|_m^2 dr \right. \\
&\quad \left. + \int_s^T \|\phi_{nk}(r)\|_{m-1}^2 dr + \alpha^2 \int_s^T (\|\theta_{Y^{nk}}(r)\|_m^2 \|\theta_{Y^n}(r)\|_m^2 + \|\theta_{Y^{nk}}(r)\|_m^2 \|\theta_{Y^k}(r)\|_m^2) dr \right\} \\
&\leq \|\psi_{nk}\|_m^2 + \frac{\nu}{2} \int_s^T (\|\theta_{Z^{nk}}(r)\|_m^2 + \|\theta_{Y^{nk}}(r)\|_m^2) dr \\
&\quad + C \left(\int_0^T (\|b_{nk}(r)\|_m^2 + \|\phi_{nk}(r)\|_{m-1}^2) dr + \int_s^T \|\theta_{Y^{nk}}(r)\|_m^2 dr \right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \sup_{s \in [T_0 + \varepsilon, T]} \|\theta_{Y^{nk}}(s)\|_m^2 + \nu \int_{T_0 + \varepsilon}^T \|\theta_{Z^{nk}}(r)\|_m^2 dr \\
&\leq C \left(\|\psi_{nk}\|_m^2 + \int_0^T (\|b_{nk}(r)\|_m^2 + \|\phi_{nk}(r)\|_{m-1}^2) dr \right) \longrightarrow 0 \text{ as } n, k \rightarrow \infty,
\end{aligned} \tag{4.14}$$

where the constant C is independent of n and k . Define $\theta_{Z^n} := \nabla \theta_{Y^n}$ for $n \in \mathbb{Z}^+$. Then $\{(\theta_{Y^n}, \theta_{Z^n}), n \in \mathbb{Z}^+\}$ is a Cauchy sequence in $C([T_0 + \varepsilon, T]; H^m) \times L^2(T_0 + \varepsilon, T; H^m)$, whose limit is denoted by $(\zeta, \nabla \zeta)$. By taking limits and in view of the arbitrariness of ε , we check that the pair $(\zeta, \nabla \zeta)$ satisfies the energy equality (4.11) on $(T_0, T]$ with (θ_Y, θ_Z) being replaced with $(\zeta, \nabla \zeta)$. Moreover, through a bootstrap argument, we can extend the existing interval to a maximal one still denoted by $(T_0, T]$ with T_0 depending on $\|\psi\|_m, \|b\|_{C([0, T]; H^m)}, \|\phi\|_{L^2(0, T; H^{m-1})}, \alpha, \nu$ and T .

On the other hand, consider the following FBSDE:

$$\begin{cases} dX_s(t, x) = [b(s, X_s(t, x)) + \alpha \zeta(s, X_s(t, x))] ds + \sqrt{\nu} dW_s, & s \in [t, T]; \\ X_t(t, x) = x; \\ -dY_s(t, x) = \phi(s, X_s(t, x)) ds - \sqrt{\nu} Z_s(t, x) dW_s, & s \in [t, T]; \\ Y_T(t, x) = \psi(X_T(t, x)), \end{cases} \tag{4.15}$$

which admits a unique local solution (X, Y, Z) on $(T_0, T]$. From Lemma 4.2, we have (4.2), (4.3) and (4.4). Letting $k \rightarrow \infty$ in (4.14), we have $\zeta(t, x) = \theta_Y(t, x)$ and $\nabla \zeta(t, x) = \theta_Z(t, x)$ for *a.e.* $(t, x) \in (T_0, T] \times \mathbb{R}^d$. Furthermore, from Lemma 4.2, we deduce that the triple $(X_s(t, x), \zeta(s, X_s(t, x)), \nabla \zeta(s, X_s(t, x)))$ solves FBSDE (4.1) and satisfies all the assertions of this proposition except the uniqueness, which is left to the next step.

Step 2. We now verify the uniqueness. Let (X, Y, Z) be any locally bounded solution of (4.1) on $(T_0, T]$. For each $t \in (T_0, T]$ and almost every $x \in \mathbb{R}^d$, define the following equivalent probability measure:

$$d\mathbb{Q}^{t, x} := \exp \left(-\frac{1}{\sqrt{\nu}} \int_t^T [b(s, X_s(t, x)) + \alpha Y_s(t, x)] dW_s - \frac{1}{2\nu} \int_t^T |b(s, X_s(t, x)) + \alpha Y_s(t, x)|^2 ds \right) d\mathbb{P}.$$

Then FBSDE (4.1) reads

$$\begin{cases} dX_s(t, x) = \sqrt{\nu} dW'_s, & s \in [t, T]; \\ X_t(t, x) = x; \\ -dY_s(t, x) = \left[\phi(s, X_s(t, x)) + Z_s(t, x)(b(s, X_s(t, x)) + \alpha Y_s(t, x)) \right] ds - \sqrt{\nu} Z_s(t, x) dW'_s, & s \in [t, T]; \\ Y_T(t, x) = \psi(X_T(t, x)), \end{cases}$$

where $(W', \mathbb{Q}^{t,x})$ is a standard Brownian motion.

Define

$$\tilde{Y}_s^n(t, \cdot) = \theta_{Y^n}(s, X_s(t, \cdot)) \text{ and } \tilde{Z}_s^n(t, \cdot) = \theta_{Z^n}(s, X_s(t, \cdot)). \quad (4.16)$$

As $m > d/2$ and $H^m \hookrightarrow C^{0,\delta}(\mathbb{R}^d)$ for some $\delta \in (0, 1)$, there is a constant N^t such that

$$\sup_n \left(\sup_{s \in [t, T], x \in \mathbb{R}^d} |\theta_{Y^n}(s, x)| + \int_t^T \sup_{x \in \mathbb{R}^d} |\theta_{Z^n}(s, x)| ds \right) \leq N^t.$$

By Remark 4.2, we have for almost all $x \in \mathbb{R}^d$,

$$\begin{aligned} \tilde{Y}_s^n(t, x) &= \psi_n(X_T(t, x)) - \sqrt{\nu} \int_s^T \tilde{Z}_r^n(t, x) dW'_r \\ &\quad + \int_s^T [\tilde{Z}_r^n(t, x)(b_n(r, X_r(t, x)) + \alpha \tilde{Y}_r^n(t, x)) + \phi_n(r, X_r(t, x))] dr, \quad t \leq s \leq T. \end{aligned}$$

Then Itô's formula yields

$$\begin{aligned} &|\tilde{Y}_s^n(t, x) - Y_s(t, x)|^2 + \nu \int_s^T |\tilde{Z}_r^n(t, x) - Z_r(t, x)|^2 dr \\ &= |\psi_n(X_T(t, x)) - \psi(X_T(t, x))|^2 + 2 \int_s^T \langle \tilde{Y}_r^n(t, x) - Y_r(t, x), \\ &\quad \tilde{Z}_r^n(t, x)(b_n(r, X_r(t, x)) + \alpha \tilde{Y}_r^n(t, x)) - Z_r(t, x)(b(r, X_r(t, x)) + \alpha Y_r(t, x)) \\ &\quad + \phi_n(r, X_r(t, x)) - \phi(r, X_r(t, x)) \rangle dr - 2 \int_s^T \langle \tilde{Y}_r^n(t, x) - Y_r(t, x), (\tilde{Z}_r^n - Z_r)(t, x) dW'_r \rangle. \end{aligned}$$

Note that both $(\tilde{Y}^n(t, x), \tilde{Z}^n(t, x))$ and $((Y(t, x), Z(t, x)))$ belong to $S^2([t, T]; \mathbb{R}^d) \times L^2_{\mathcal{F}}(t, T; \mathbb{R}^d \times \mathbb{R}^d)$ and moreover, there exists a constant $K^{t,x}$ such that $\sup_{s \in [t, T]} |Y_s(t, x)| \leq K^{t,x}$, a.s.. Then, we have

$$\begin{aligned} &E_{\mathbb{Q}^{t,x}} \left[|\tilde{Y}_s^n(t, x) - Y_s(t, x)|^2 + \nu \int_s^T |\tilde{Z}_r^n(t, x) - Z_r(t, x)|^2 dr \right] \\ &= E_{\mathbb{Q}^{t,x}} [|\psi_n(X_T(t, x)) - \psi(X_T(t, x))|^2] + 2E_{\mathbb{Q}^{t,x}} \left[\int_s^T \langle \tilde{Y}_r^n(t, x) - Y_r(t, x), \right. \\ &\quad \tilde{Z}_r^n(t, x)(b_n(r, X_r(t, x)) + \alpha \tilde{Y}_r^n(t, x)) - Z_r(t, x)(b(r, X_r(t, x)) + \alpha Y_r(t, x)) + \phi_n(r, X_r(t, x)) \\ &\quad \left. - \phi(r, X_r(t, x)) \rangle dr \right] \\ &\leq \frac{\nu}{2} E_{\mathbb{Q}^{t,x}} \left[\int_s^T |\tilde{Z}_r^n(t, x) - Z_r(t, x)|^2 dr \right] + C(m, d) \|\psi_n - \psi\|_m^2 \\ &\quad + C(\nu, \alpha) E_{\mathbb{Q}^{t,x}} \left[\int_s^T (|\phi_n(r, X_r(t, x)) - \phi(r, X_r(t, x))|^2 + N^t \|b_n - b\|_{C([0, T]; H^m)}^2 \right. \\ &\quad \left. + |Y_r^n(t, x) - Y_r(t, x)|^2 (1 + \|b(r)\|_m^2 + \|\theta_{Z^n}(r)\|_{C(\mathbb{R}^d)} + |Y_r(t, x)|^2)) dr \right] \\ &\leq \frac{\nu}{2} E_{\mathbb{Q}^{t,x}} \left[\int_s^T |\tilde{Z}_r^n(t, x) - Z_r(t, x)|^2 dr \right] + C(m, d) \|\psi_n - \psi\|_m^2 \\ &\quad + CE_{\mathbb{Q}^{t,x}} \left[\int_s^T (|\phi_n(r, X_r(t, x)) - \phi(r, X_r(t, x))|^2 + \|b_n - b\|_{C([0, T]; H^m)}^2 \right. \\ &\quad \left. + |Y_r^n(t, x) - Y_r(t, x)|^2 (1 + \|\theta_{Z^n}(r)\|_{C(\mathbb{R}^d)})^2) dr \right] \end{aligned}$$

Then, from Gronwall Inequality, we have

$$\begin{aligned} & \sup_{s \in [t, T]} E_{\mathbb{Q}^{t, x}} \left[|\tilde{Y}_s^n(t, x) - Y_s(t, x)|^2 \right] + \nu E_{\mathbb{Q}^{t, x}} \left[\int_t^T |\tilde{Z}_s^n(t, x) - Z_s(t, x)|^2 ds \right] \\ & \leq C \left\{ \|\psi_n - \psi\|_m^2 + \|b - b_n\|_{C([0, T]; H^m)}^2 + E_{\mathbb{Q}^{t, x}} \left[\int_s^T (|\phi_n(r, X_r(t, x)) - \phi(r, X_r(t, x))|^2) dr \right] \right\} \longrightarrow 0, \end{aligned}$$

where the constant C depends only on $N^t, K^{t, x}, T, \|b\|_{C([0, T]; H^m)}, T, m, d, \nu$ and α , and is independent of n .

Thus, in view of (4.16), we conclude that

$$Y_s(t, x) = \zeta(s, X_s(t, x)) \text{ and } Z_s(t, x) = \nabla \zeta(s, X_s(t, x)).$$

Therefore, any locally bounded solution of FBSDE (4.1) on $(T_0, T]$ must have the form described as above. Now, let (X, Y, Z) and $(\bar{X}, \bar{Y}, \bar{Z})$ be any two locally bounded solutions of FBSDE (4.1) on $(T_0, T]$. By previous argument we have

$$\begin{aligned} Y_s(t, x) &= \zeta(s, X_s(t, x)), \quad Z_s(t, x) = \nabla \zeta(s, X_s(t, x)), \\ \bar{Y}_s(t, x) &= \zeta(s, X_s(t, x)), \quad \bar{Z}_s(t, x) = \nabla \zeta(s, X_s(t, x)). \end{aligned} \tag{4.17}$$

Hence $X_s(t, x)$ and $\bar{X}_s(t, x)$ satisfy the same forward SDE with the same initial value. Thus we must have $X \equiv \bar{X}$, a.s., which in turn shows that $(Y, Z) = (\bar{Y}, \bar{Z})$, a.s..

Step 3. For $m > d/2 + 1$, to prove that the unique locally bounded solution on $(T_0, T]$ is also the unique local solution, it is sufficient to prove that the locally bounded solution constructed in **Step 1**

$$(X_s(t, x), \zeta(s, X_s(t, x)), \nabla \zeta(s, X_s(t, x)))_{T_0 < t \leq s \leq T}$$

is the unique local solution to FBSDE (4.1) on $(T_0, T]$ as well.

Since $m > d/2 + 1$ and $H^{m-1} \hookrightarrow C^{0, \delta}(\mathbb{R}^d)$ for some $\delta \in (0, 1)$, our BSDE is well-posed for each $x \in \mathbb{R}^d$. Let (X, Y, Z) be any solution of (4.1) on $[t, T]$ with $t \in (T_0, T]$. For every $x \in \mathbb{R}^d$, define

$$\tilde{Y}_s^n(t, x) = \theta_{Y^n}(s, X_s(t, x)) \text{ and } \tilde{Z}_s^n(t, x) = \theta_{Z^n}(s, X_s(t, x)). \tag{4.18}$$

By Remark 4.2, we have

$$\begin{aligned} \tilde{Y}_s^n(t, x) &= \psi_n(X_T(t, x)) - \sqrt{\nu} \int_s^T \tilde{Z}_r^n(t, x) dW_r \\ &\quad + \int_s^T [\tilde{Z}_r^n(t, x)(b_n(r, X_r(t, x)) + \alpha \tilde{Y}_r^n(t, x) - b(r, X_r(t, x)) - \alpha Y_r(t, x)) \\ &\quad + \phi_n(r, X_r(t, x))] dr, \quad t \leq s \leq T, \text{ a.e. } x \in \mathbb{R}^d. \end{aligned}$$

Then Itô's formula yields

$$\begin{aligned} & |\tilde{Y}_s^n(t, x) - Y_s(t, x)|^2 + \nu \int_s^T |\tilde{Z}_r^n(t, x) - Z_r(t, x)|^2 dr \\ &= |\psi_n(X_T(t, x)) - \psi(X_T(t, x))|^2 + 2 \int_s^T \langle \tilde{Y}_r^n(t, x) - Y_r(t, x), \\ &\quad \tilde{Z}_r^n(t, x)(b_n(r, X_r(t, x)) + \alpha \tilde{Y}_r^n(t, x) - b(r, X_r(t, x)) - \alpha Y_r(t, x)) \\ &\quad + \phi_n(r, X_r(t, x)) - \phi(r, X_r(t, x)) \rangle dr - 2 \int_s^T \langle \tilde{Y}_r^n(t, x) - Y_r(t, x), (\tilde{Z}_r^n - Z_r)(t, x) dW_r \rangle. \end{aligned}$$

Note that both $(\tilde{Y}_s^n(t, x), \tilde{Z}_s^n(t, x))$ and $((Y_s(t, x), Z_s(t, x)))$ belong to $S^2([t, T]; \mathbb{R}^d) \times L^2_{\mathcal{F}}(t, T; \mathbb{R}^d \times \mathbb{R}^d)$ and moreover that there exists a positive constant K_t such that

$$\sup_n \sup_{s \in [t, T]} |\tilde{Z}_s^n(t, x)| \leq C \|\theta_{Z^n}\|_{C([t, T]; H^{m-1})} \leq K_t, \text{ a.s., } \forall x \in \mathbb{R}^d.$$

Thus, we have

$$\begin{aligned}
& E \left[|\tilde{Y}_s^n(t, x) - Y_s(t, x)|^2 + \nu \int_s^T |\tilde{Z}_r^n(t, x) - Z_r(t, x)|^2 dr \right] \\
&= E[|\psi_n(X_T(t, x)) - \psi(X_T(t, x))|^2] + 2E \left[\int_s^T \langle \tilde{Y}_r^n(t, x) - Y_r(t, x), \right. \\
&\quad \tilde{Z}_r^n(t, x)(b_n(r, X_r(t, x)) + \alpha \tilde{Y}_r^n(t, x) - b(r, X_r(t, x)) - \alpha Y_r(t, x)) + \phi_n(r, X_r(t, x)) \\
&\quad \left. - \phi(r, X_r(t, x)) \rangle dr \right] \\
&\leq C(K_t, T, \alpha, \nu, d, m) E \left[\int_s^T |\tilde{Y}_r^n(t, x) - Y_r(t, x)| (|\tilde{Y}_r^n(t, x) - Y_r(t, x)| + \|b(r) - b_n(r)\|_m \right. \\
&\quad \left. + \|\phi_n(r) - \phi(r)\|_{m-1}) dr \right] + C\|\psi_n - \psi\|_m^2.
\end{aligned}$$

Using Gronwall inequality, we have

$$\begin{aligned}
& \sup_{s \in [t, T]} E \left[|\tilde{Y}_s^n(t, x) - Y_s(t, x)|^2 \right] + \nu E \left[\int_t^T |\tilde{Z}_s^n(t, x) - Z_s(t, x)|^2 ds \right] \\
&\leq C\{\|\psi_n - \psi\|_m^2 + \|b - b_n\|_{C([0, T]; H^m)}^2 + \|\phi_n - \phi\|_{L^2(0, T; H^{m-1})}^2\} \longrightarrow 0.
\end{aligned} \tag{4.19}$$

Thus, in view of (4.18), we conclude that

$$Y_s(t, x) = \zeta(s, X_s(t, x)) \text{ and } Z_s(t, x) = \nabla \zeta(s, X_s(t, x)).$$

Analogous to the arguments of **Step 2**, we verify that the triple constructed in **Step 1**

$$(X_s(t, x), \zeta(s, X_s(t, x)), \nabla \zeta(s, X_s(t, x)))_{T_0 < t \leq s \leq T}$$

is the unique local solution to FBSDE (4.1) on $(T_0, T]$ as well. The proof is complete. \square

Remark 4.3. In Step 2, the equation (4.3) plays a crucial role in the proof of uniqueness. Indeed, FBSDE (4.1) is usually associated to the deterministic vector field θ_Y which satisfies (4.3) instead of being a classical solution to some parabolic PDE in [29]. Note that equation (4.3) is probabilistic and that according to Lemma 3.1, it makes sense for our Sobolev coefficients. Therefore, our method here helps to probabilistically solve more general coupled FBSDEs.

On the other hand, in view of the whole proof of Proposition 4.3, we have

$$T_0 \leq 0 \vee \left[T - \frac{1}{\alpha^2 C_0 C(\nu, b)} \right]$$

and also, if $T\alpha^2 C_0 C(\nu, b) < 1$ in (4.13), then the local bounded solution is actually a global solution on the whole interval $[0, T]$.

Remark 4.4. If $\tilde{Y}_0(\cdot, \cdot)$ of FBSDS (1.3) lies in $L^2(T_0, T; H^{m-1})$, then θ_Y of (1.3) is deterministic and belongs to $L^2(T_0, T; H^{m+1})$. Therefore, by Lemma 3.1 and Proposition 4.3, Definition 3.2 and Remark 3.2 make senses.

From Proposition 4.3, we have the following characterization of an H^m -solution to FBSDS (1.3) for $m > d/2$, whose proof is omitted.

Corollary 4.4. *Under assumptions of Theorem 3.3, (X, Y, Z, \tilde{Y}_0) is an (local, respectively) H^m -solution of FBSDS (1.3) (on some time interval $(T_0, T]$, respectively) if and only if (X, Y, Z, \tilde{Y}_0) is a solution of FBSDS (1.3) with $\theta_Y \in C([0, T]; H^m)$ ($\theta_Y \in C_{loc}((T_0, T]; H^m)$, respectively).*

4.2 Proof of Theorem 3.3

First, for each $v \in C([0, T]; H_\sigma^m) \cap L^2(0, T; H^{m+1})$, $\zeta \in C([0, T]; H^m) \cap L^2(0, T; H^{m+1})$, consider the following FBSDS:

$$\left\{ \begin{array}{l} dX_s(t, x) = v(s, X_s(t, x)) ds + \sqrt{\nu} dW_s, \quad s \in [t, T]; \\ X_t(t, x) = x; \\ -dY_s(t, x) = \left[f(s, X_s(t, x)) + \tilde{Y}_0(s, X_s(t, x)) \right] ds - \sqrt{\nu} Z_s(t, x) dW_s, \quad s \in [t, T]; \\ Y_T(t, x) = G(X_T(t, x)); \\ -d\tilde{Y}_s(t, x) = \frac{27}{2s^3} v^i \zeta^j(t, x + B_s) \left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i \right) \left(B_s^j - B_{\frac{2s}{3}}^j \right) B_{\frac{s}{3}} ds \\ \quad - dM_s, \quad s \in (0, \infty); \\ \tilde{Y}_\infty(t, x) = 0. \end{array} \right. \quad (4.20)$$

By BSDE theory (see [25, 33]) and Lemma 3.2, FBSDS (4.20) admits a unique solution $(X^{v, \zeta}, Y^{v, \zeta}, Z^{v, \zeta}, \tilde{Y}_0^{v, \zeta})$ and in view of Lemma 3.2 and Remark 3.3, there holds that

$$Y_0(t, x) = -\mathbf{P}^\perp(\operatorname{div}(v(t, x) \otimes \zeta(t, x))) = -\mathbf{P}^\perp((v(t, x) \cdot \nabla) \zeta(t, x)),$$

where we have used the fact that $\operatorname{div}(v) = 0$. From Proposition 4.3 and Remark 4.3, we conclude that $(\theta_{Y^{v, \zeta}}, \theta_{Z^{v, \zeta}}) \in C([0, T]; H^m) \times L^2(0, T; H^m)$.

For any $\zeta_i \in C([0, T]; H^m) \cap L^2(0, T; H^{m+1})$, $i = 1, 2$, set

$$(\delta\theta_{Y^{v, \zeta}}, \delta\theta_{Z^{v, \zeta}}, \delta\zeta) := (\theta_{Y^{v, \zeta_1}} - \theta_{Y^{v, \zeta_2}}, \theta_{Z^{v, \zeta_1}} - \theta_{Z^{v, \zeta_2}}, \zeta_1 - \zeta_2).$$

Then by Proposition 4.3, we have

$$\begin{aligned} & \|\delta\theta_{Y^{v, \zeta}}(s)\|_m^2 + \nu \int_s^T \|\delta\theta_{Z^{v, \zeta}}(r)\|_m^2 dr \\ &= \int_s^T 2 \langle \delta\theta_{Z^{v, \zeta}} v(r), \delta\theta_{Y^{v, \zeta}}(r) \rangle_{m-1, m+1} dr - \int_s^T 2 \langle \mathbf{P}^\perp((v \cdot \nabla) \delta\zeta)(r), \delta\theta_{Y^{v, \zeta}}(r) \rangle_{m-1, m+1} dr \\ &\leq \frac{\nu}{2} \int_s^T (\|\delta\theta_{Y^{v, \zeta}}(r)\|_m^2 + \|\delta\theta_{Z^{v, \zeta}}(r)\|_m^2) dr \\ &\quad + C(\nu) \left(\int_s^T \|v(r)\|_m^2 \|\delta\theta_{Y^{v, \zeta}}(r)\|_m^2 dr + \int_s^T \|v(r)\|_m^2 \|\delta\zeta(r)\|_m^2 dr \right) \end{aligned}$$

Using Gronwall inequality, we obtain

$$\sup_{s \in [t, T]} \|\delta\theta_{Y^{v, \zeta}}(s)\|_m^2 + \int_t^T \|\delta\theta_{Z^{v, \zeta}}(r)\|_m^2 dr \leq C(T-t) \|\delta\zeta\|_{C([t, T]; H^m)}^2 \quad (4.21)$$

with the constant C depending on $\nu, \|v\|_{C([0, T]; H^m)}$ and T . Then, by the contraction mapping principle we can choose a small enough positive constant $\epsilon \leq T$ depending only on $\nu, \|v\|_{C([0, T]; H^m)}$ and T , such that there exists a unique function $\bar{\zeta} \in C([T-\epsilon, T]; H^m)$ satisfying

$$(\theta_{Y^{v, \bar{\zeta}}}, \theta_{Z^{v, \bar{\zeta}}}) = (\bar{\zeta}, \nabla \bar{\zeta}) \text{ in } C([T-\epsilon, T]; H^m) \times L^2(T-\epsilon, T; H^m).$$

Then by Lemmas 3.2 and 4.2, we have for almost all $x \in \mathbb{R}^d$,

$$\begin{aligned} \theta_{Z^{v, \bar{\zeta}}}(t, x) &= \nabla \theta_{Y^{v, \bar{\zeta}}}(t, x), \quad Y_r^{v, \bar{\zeta}}(t, x) = \theta_{Y^{v, \bar{\zeta}}}(r, X_r(t, x)), \quad Z_r^{v, \bar{\zeta}}(t, x) = \theta_{Z^{v, \bar{\zeta}}}(r, X_r(t, x)), \text{ a.s.}, \\ \theta_{Y^{v, \bar{\zeta}}}(r, X_r(t, x)) &= G(X_T(t, x)) + \int_r^T \left[f(s, X_s(t, x)) - \mathbf{P}^\perp((v \cdot \nabla) \theta_{Y^{v, \bar{\zeta}}})(s, X_s(t, x)) \right] ds \\ &\quad - \sqrt{\nu} \int_r^T \theta_{Z^{v, \bar{\zeta}}}(s, X_s(t, x)) dW_s, \text{ a.s.} \end{aligned}$$

For each $(t, x) \in [T - \varepsilon, T) \times \mathbb{R}^d$, define the following equivalent probability measure:

$$d\mathbb{Q}^{t,x} := \exp \left(-\frac{1}{\sqrt{\nu}} \int_t^T v(s, X_s(t, x)) dW_s - \frac{1}{2\nu} \int_t^T |b(s, X_s(t, x))|^2 ds \right) d\mathbb{P}.$$

Then FBSDS (4.20) reads

$$\left\{ \begin{array}{l} dX_s(t, x) = \sqrt{\nu} dW'_s, \quad s \in [t, T]; \\ X_t(t, x) = x; \\ -dY_s(t, x) = \left[f + \theta_{Z^v} + \tilde{Y}_0 \right] (s, X_s(t, x)) ds - \sqrt{\nu} Z_s(t, x) dW'_s, \quad s \in [t, T]; \\ Y_T(t, x) = G(X_T(t, x)); \\ -d\tilde{Y}_s(t, x) = \frac{27}{2s^3} v^i \theta_Y^j(t, x + B_s) \left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i \right) \left(B_s^j - B_{\frac{2s}{3}}^j \right) B_{\frac{s}{3}} ds \\ \quad - dM_s, \quad s \in (0, \infty); \\ \tilde{Y}_\infty(t, x) = 0, \end{array} \right. \quad (4.22)$$

where $(W', \mathbb{Q}^{t,x})$ is a standard Brownian motion. Then taking the divergence operator on both sides of the above third and fourth equalities, we conclude that

$$\operatorname{div} \theta_{Y^v, \tilde{\zeta}}(s, y) = 0, a.s. \forall s \in (t, T], a.e. y \in \mathbb{R}^d.$$

On the other hand, by Proposition 4.3 and Lemma 2.2, we have

$$\begin{aligned} & \|\theta_{Y^v, \tilde{\zeta}}(s)\|_m^2 + \nu \int_s^T \|\theta_{Z^v, \tilde{\zeta}}(r)\|_m^2 dr \\ &= \|G\|_m^2 + \int_s^T 2 \langle \mathbf{P}((v \cdot \nabla) \theta_{Y^v, \tilde{\zeta}})(r), \theta_{Y^v, \tilde{\zeta}}(r) \rangle_{m-1, m+1} dr + \int_s^T 2 \langle f(r), \theta_{Y^v, \tilde{\zeta}}(r) \rangle_{m-1, m+1} dr \\ &\leq \|G\|_m^2 + C \left(\int_s^T \|v(r)\|_m \|\theta_{Y^v, \tilde{\zeta}}(r)\|_m^2 dr + \frac{1}{\nu} \int_s^T \|f(r)\|_{m-1}^2 dr \right) \\ &\quad + \frac{\nu}{2} \int_s^T (\|\theta_{Y^v, \tilde{\zeta}}(r)\|_m^2 + \|\theta_{Z^v, \tilde{\zeta}}(r)\|_m^2) dr \end{aligned}$$

which together with the Gronwall inequality implies

$$\begin{aligned} & \sup_{s \in [t, T]} \|\theta_{Y^v, \tilde{\zeta}}(s)\|_m^2 + \frac{\nu}{2} \int_t^T \|\theta_{Z^v, \tilde{\zeta}}(r)\|_m^2 dr \\ &\leq C(\nu, T) \left(\|f\|_{L^2(0, T; H^{m-1})}^2 + \|G\|_m^2 \right) e^{(\|v\|_{C([t, T]; H^m)} + \nu)(T-t)}. \end{aligned} \quad (4.23)$$

Hence, through a bootstrap argument, we conclude that there exists a unique function $\tilde{\zeta} \in C([0, T]; H_\sigma^m)$ satisfying $(\theta_{Y^v, \tilde{\zeta}}, \theta_{Z^v, \tilde{\zeta}}) = (\tilde{\zeta}, \nabla \tilde{\zeta})$ in $C([0, T]; H_\sigma^m) \times L^2(0, T; H_\sigma^m)$, and again by Lemma 4.2, we conclude that

$$(X^v, Y^v, Z^v, \tilde{Y}_0^v) := (X^{v, \tilde{\zeta}}, Y^{v, \tilde{\zeta}}, Z^{v, \tilde{\zeta}}, \tilde{Y}_0^{v, \tilde{\zeta}})$$

is the unique H^m -solution of the following FBSDS:

$$\left\{ \begin{array}{l} dX_s(t, x) = v(s, X_s(t, x)) ds + \sqrt{\nu} dW_s, \quad s \in [t, T]; \\ X_t(t, x) = x; \\ -dY_s(t, x) = \left[f(s, X_s(t, x)) + \tilde{Y}_0(s, X_s(t, x)) \right] ds - \sqrt{\nu} Z_s(t, x) dW_s, \quad s \in [t, T]; \\ Y_T(t, x) = G(X_T(t, x)); \\ -d\tilde{Y}_s(t, x) = \frac{27}{2} v^i Y_t^j(t, x + B_s) \left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i \right) \left(B_s^j - B_{\frac{2s}{3}}^j \right) B_{\frac{s}{3}} s^{-3} ds \\ \quad - dM_s, \quad s \in (0, \infty); \\ \tilde{Y}_\infty(t, x) = 0. \end{array} \right. \quad (4.24)$$

Choose two positive real numbers R and ε ($\varepsilon < T$) whose values are to be determined later, and define

$$U_R^\varepsilon := \left\{ u \in C([T-\varepsilon, T]; H_\sigma^m) \cap L^2(T-\varepsilon, T; H_\sigma^{m+1}) : \right. \\ \left. \|u\|_{C([T-\varepsilon, T]; H_\sigma^m)}^2 + \frac{\nu}{2} \|\nabla u\|_{L^2(T-\varepsilon, T; H_\sigma^{m+1})}^2 \leq R^2 \right\}.$$

For any $v \in U_R^\varepsilon$, there holds the following estimate by (4.23):

$$\sup_{s \in [T-\varepsilon, T]} \|\theta_{Y^v}(s)\|_m^2 + \frac{\nu}{2} \int_{T-\varepsilon}^T \|\theta_{Z^v}(s)\|_m^2 dr \leq C(\nu, \|f\|_{L^2(0, T; H^{m-1})}^2, \|G\|_m^2, T) e^{R\varepsilon}. \quad (4.25)$$

Choosing R to be big enough and ε to be small enough, we have

$$\sup_{s \in [T-\varepsilon, T]} \|\theta_{Y^v}(s)\|_m^2 + \frac{\nu}{2} \int_{T-\varepsilon}^T \|\theta_{Z^v}(r)\|_m^2 dr \leq R^2.$$

On the other hand, for any $v_1, v_2 \in U_R^\varepsilon$, setting

$$(\delta\theta_{Y^v}, \delta\theta_{Z^v}, \delta v) := (\theta_{Y^{v_1}} - \theta_{Y^{v_2}}, \theta_{Z^{v_1}} - \theta_{Z^{v_2}}, v_1 - v_2),$$

we have

$$\begin{aligned} & \|\delta\theta_{Y^v}(s)\|_m^2 + \nu \int_s^T \|\delta\theta_{Z^v}(r)\|_m^2 dr \\ &= 2 \int_s^T \langle \mathbf{P}((\delta v \cdot \nabla) \theta_{Y^{v_1}})(r), \delta\theta_{Y^v}(r) \rangle_{m-1, m+1} dr + 2 \int_s^T \langle \mathbf{P}((v_2 \cdot \nabla) \delta\theta_{Y^v})(r), \delta\theta_{Y^v}(r) \rangle_{m-1, m+1} dr \\ &\leq \frac{\nu}{2} \int_s^T (\|\delta\theta_{Y^v}(r)\|_m^2 + \|\delta\theta_{Z^v}(r)\|_m^2) dr \\ &\quad + C(\nu) \left(\int_s^T \|\delta v(r)\|_m^2 \|\theta_{Y^{v_1}}(r)\|_m^2 dr + \int_s^T \|v_2(r)\|_m^2 \|\delta\theta_{Y^v}(r)\|_m^2 dr \right) \end{aligned}$$

which together with the Gronwall-Bellman inequality, implies

$$\sup_{s \in [T-\varepsilon, T]} \|\delta\theta_{Y^v}(s)\|_m^2 + \frac{\nu}{2} \int_{T-\varepsilon}^T \|\delta\theta_{Z^v}(r)\|_m^2 dr \leq C(\nu) R^2 e^{R^2 T} \varepsilon \|\delta v\|_{C([T-\varepsilon, T]; H_\sigma^m)}^2.$$

Therefore, if we choose ε to be small enough, the solution map $\Psi : v \mapsto \theta_{Y^v}$ is a contraction mapping on the complete metric space U_R^ε and then through a bootstrap argument, we obtain a unique function $\bar{u} \in C_{loc}((T_0, T]; H_\sigma^m) \cap L_{loc}^2(T_0, T; H_\sigma^{m+1})$ satisfying $(\theta_{Y^{\bar{u}}}, \theta_{Z^{\bar{u}}}) = (\bar{u}, \nabla \bar{u})$ on $(T_0, T] \times \mathbb{R}^d$ with T_0 depending on $\nu, T, \|G\|_m$ and $\|f\|_{L^2(0, T; H^{m-1})}$. From Proposition 4.3 and the contraction mapping principle, we have

$$(X, Y, Z, \tilde{Y}_0) := (X^{\bar{u}}, Y^{\bar{u}}, Z^{\bar{u}}, \tilde{Y}_0^{\bar{u}})$$

is the unique local H^m -solution of FBSDS (1.3) and all the other assertions hold as well.

From Remarks 3.3 and 3.4, we deduce that there exists some $p \in L_{loc}^2(T_0, T; H_\sigma^m)$ such that $\tilde{Y}_0 = \nabla p$. For each $t \in (T_0, T]$ a.e. $x \in \mathbb{R}^d$, define the following equivalent probability $Q^{t, x}$:

$$d\mathbb{Q}^{t, x} := \exp \left(-\frac{1}{\sqrt{\nu}} \int_t^T \theta_Y(s, X_s(t, x)) dW_s - \frac{1}{2\nu} \int_t^T |b(s, X_s(t, x))|^2 ds \right) d\mathbb{P}.$$

Then we have

$$\left\{ \begin{array}{l} dX_s(t, x) = \sqrt{\nu} dW'_s, \quad s \in [t, T]; \\ X_t(t, x) = x; \\ -dY_s(t, x) = [(\theta_Y \cdot \nabla) \theta_Y + f + \nabla p](s, X_s(t, x)) ds - \sqrt{\nu} \nabla \theta_Y(s, X_s(t, x)) dW'_s, \quad s \in [t, T]; \\ Y_T(t, x) = G(X_T(t, x)), \end{array} \right.$$

where $(W', \mathbb{Q}^{t,x})$ is a standard Brownian motion. For any $\zeta \in C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathcal{O})$, Itô's formula yields that

$$\zeta(s, X_s(t, x)) = \zeta(T, X_T(t, x)) - \int_s^T (\partial_r + \frac{\nu}{2} \Delta) \zeta(r, X_r(t, x)) dr - \int_s^T \nabla \zeta(r, X_r(t, x)) dW'_r$$

and thus,

$$\begin{aligned} & E_{\mathbb{Q}^{t,x}}[\langle \theta_Y, \zeta \rangle(t, x)] + \nu E \left[\int_t^T \langle \nabla \zeta, \nabla \theta_Y \rangle(s, X_s(t, x)) ds \right] \\ &= E_{\mathbb{Q}^{t,x}} \left[\langle \zeta(T, X_T(t, x)), G(x_T(t, x)) \rangle \right. \\ & \quad \left. + \int_t^T (\langle -\partial_s \zeta - \frac{\nu}{2} \Delta \zeta, \theta_Y \rangle + \langle \zeta, (\theta \cdot \nabla) \theta + \nabla p + f \rangle)(s, X_s(t, x)) ds \right]. \end{aligned}$$

Integrating both sides of the last equality with respect to x , we have

$$\langle \zeta(t), \theta_Y(t) \rangle_0 = \langle \zeta(T), G \rangle_0 + \int_t^T \left[-\langle \partial_s \zeta(s), \theta_Y(s) \rangle_0 + \langle \zeta(s), \frac{\nu}{2} \Delta \theta_Y(s) + (\theta_Y \cdot \nabla) \theta_Y(s) \rangle_0 \right] ds$$

Hence, (θ_Y, p) is a strong solution to Navier-Stokes equation (3.16) (see [40, 41]). Because of the reversibility of the above procedure and the uniqueness of the H^m -solution of FBSDS (1.3), we prove the uniqueness of the strong solution for Navier-Stokes equation (3.16) as well.

Remark 4.5. Proposition 4.3 together with the contraction mapping principle serves to guarantee the existence and uniqueness of the local H^m -solution of FBSDS (1.3). Furthermore, in a similar way to the above proof, we can prove that θ_Y of Proposition 4.3 is in fact the unique local strong solution of the following PDE

$$\begin{cases} \partial_t u + \frac{\nu}{2} \Delta u + ((b + \alpha u) \cdot \nabla) u + \phi = 0, & t \leq T; \\ u(T) = \psi, \end{cases} \quad (4.26)$$

which is the well-known Burgers equation if $\alpha = 1$ and $b \equiv 0$.

5 Global results

5.1 The case of small Reynolds numbers

We work on the d -dimensional torus $\mathbb{T}^d = \mathbb{R}^d / (L \times \mathbb{Z}^d)$ where $L > 0$ is a fixed length scale. Denote by $(H_\sigma^{n,q}(\mathbb{T}^d; \mathbb{R}^d), \|\cdot\|_{n,q;\mathbb{T}^d})$ the \mathbb{R}^d -valued Sobolev space on \mathbb{T}^d , each element of which is divergence free. For $q = 2$, write $(H_\sigma^n(\mathbb{T}^d; \mathbb{R}^d), \|\cdot\|_{n;\mathbb{T}^d})$ for simplicity. Let $G \in H_\sigma^m(\mathbb{T}^d; \mathbb{R}^d)$ of zero mean for $m > d/2$. Consider

$$\left\{ \begin{array}{l} dX_s(t, x) = Y_s(t, x) ds + \sqrt{\nu} dW_s, \quad s \in [t, T]; \\ X_t(t, x) = x; \\ -dY_s(t, x) = \tilde{Y}_0(s, X_s(t, x)) ds - \sqrt{\nu} Z_s(t, x) dW_s, \quad s \in [t, T]; \\ Y_T(t, x) = G(X_T(t, x)); \\ -d\tilde{Y}_s(t, x) = \frac{27}{2s^3} Y_t^i Y_t^j(t, x + B_s) \left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i \right) \left(B_s^j - B_{\frac{2s}{3}}^j \right) B_{\frac{s}{3}} ds \\ \quad - dM_s, \quad s \in (0, \infty); \\ \tilde{Y}_\infty(t, x) = 0, \end{array} \right. \quad (5.1)$$

where the external forcing is not introduced for simplicity. In a similar way to Theorem 3.3, FBSDS (5.1) admits a unique local H^m -solution on some interval $(T_0, T]$. Moreover, we have

$$\begin{aligned}
& \|\theta_Y(t)\|_{m;\mathbb{T}^d}^2 + \nu \int_t^T \|\theta_Z(s)\|_{m;\mathbb{T}^d}^2 ds \\
&= \|G\|_{m;\mathbb{T}^d}^2 + 2 \int_t^T \langle \mathbf{P}(\theta_Z(s)\theta_Y(s)), \theta_Y(s) \rangle_{m-1, m+1; \mathbb{T}^d} ds \\
&\leq \|G\|_{m;\mathbb{T}^d}^2 + C \int_t^T \|\theta_Y(s)\|_{m;\mathbb{T}^d} \|\theta_Z(s)\|_{m-1; \mathbb{T}^d} \|\theta_Y(s)\|_{m+1; \mathbb{T}^d} ds \\
&\leq \|G\|_{m;\mathbb{T}^d}^2 + \tilde{C} \int_t^T \|\theta_Y(s)\|_{m;\mathbb{T}^d} \|\theta_Z(s)\|_{m;\mathbb{T}^d}^2 ds,
\end{aligned}$$

where we have used the fact that by Poincaré inequality and the scaling properties,

$$\|\theta_Y(s, \cdot)\|_{m;\mathbb{T}^d} \leq CL \|\nabla \theta_Y(s, \cdot)\|_{m;\mathbb{T}^d}, \quad s \in [t, T],$$

with $\theta_Y(s, \cdot)$ being mean zero and the constant \tilde{C} being independent of L . Thus,

$$\|\theta_Y(t)\|_{m;\mathbb{T}^d}^2 + \int_t^T (\nu - \tilde{C}L \|\theta_Y(s)\|_{m;\mathbb{T}^d}) \|\theta_Z(s)\|_{m;\mathbb{T}^d}^2 ds \leq \|G\|_{m;\mathbb{T}^d}^2.$$

If we take the Reynolds number $R := L\nu^{-1} \|G\|_{m;\mathbb{T}^d} < \tilde{C}^{-1}$, then for this local solution (X, Y, Z, \tilde{Y}_0) we always have

$$\|\theta_Y(t)\|_{m;\mathbb{T}^d} \leq \|G\|_{m;\mathbb{T}^d}, \quad t \in (T_0, T].$$

Using bootstrap arguments, the local solution can be extended to be a global one. In summary, we have

Theorem 5.1. *Assume that $G \in H_\sigma^m(\mathbb{T}^d; \mathbb{R}^d)$ ($m > d/2$) is mean zero. FBSDS (5.1) admits one and only one local H^m -solution (X, Y, Z, \tilde{Y}_0) on some time interval $(T_0, T]$ with $\theta_Y \in H_\sigma^m(\mathbb{T}^d)$ being spacial mean zero. Moreover, there exists a positive constant $R_0 (= \frac{1}{\tilde{C}}$ as above) such that if the Reynolds number $R < R_0$, our local H^m -solution can be extended to be a time global one and for this global H^m -solution we have*

$$\|\theta_Y(t)\|_{m;\mathbb{T}^d} \leq \|G\|_{m;\mathbb{T}^d}, \quad \text{for any } t \in [0, T].$$

5.2 The two-dimensional case

Consider the two dimensional case. For simplicity, we assume $f = 0$ and $m \geq 3$. Then under the assumptions of Theorem 3.3, let (X, Y, Z, \tilde{Y}_0) be the local H^m -solution of FBSDS (1.3) on the time interval $(T_0, T]$. Define the vorticity field:

$$\vartheta_Y := \text{Curl } \theta_Y := \partial_{x_1} \theta_Y^2 - \partial_{x_2} \theta_Y^1,$$

which is scalar-valued. Consider the following FBSDE:

$$\begin{cases} dX_s(t, x) = \theta_Y(s, X_s(t, x)) ds + \sqrt{\nu} dW_s; \\ X_t(t, x) = x; \\ d\tilde{Y}_s(t, x) = \sqrt{\nu} \tilde{Z}_s dW_s; \\ \tilde{Y}_T(t, x) = (\text{Curl } G)(X_T(t, x)), \quad T_0 < t \leq s \leq T, x \in \mathbb{R}^2. \end{cases} \quad (5.2)$$

By Proposition 4.3 and Theorem 3.3, we have

$$\vartheta_Y(t, x) = \tilde{Y}_t(t, x) = E[(\text{Curl } G)(X_T(t, x))]. \quad (5.3)$$

In view of [30, page 117, Proposition 3.8], we have

$$\|\theta_Z\|_{L^\infty(\mathbb{R}^2)} \leq C(1 + \ln^+ \|\theta_Y\|_3 + \ln^+ \|\vartheta_Y\|_0)(1 + \|\vartheta_Y\|_{L^\infty(\mathbb{R}^2)}), \quad (5.4)$$

where $\ln^+ y := 0 \vee \ln y$.

On the other hand, since $\nabla \cdot \theta_Y = 0$, we have

$$\det(\nabla X_s(t, x)) = \exp \left(\int_t^s (\nabla \cdot \theta_Y)(r, X_r(t, x)) dr \right) = 1,$$

which implies

$$\|\vartheta_Y(t)\|_{L^q(\mathbb{R}^2)} \leq \|\text{Curl } G\|_{L^q(\mathbb{R}^2)} \leq C\|G\|_m, \quad q \in [2, \infty].$$

Thus, in view of (5.4), we have

$$\|\theta_Z\|_{L^\infty(\mathbb{R}^2)} \leq C(\|G\|_m)(1 + \ln^+ \|\theta_Y\|_m). \quad (5.5)$$

From the identity equation and the estimate (2.4) of Lemma 2.2, we have

$$\begin{aligned} \|\theta_Y(s)\|_m^2 + \nu \int_s^T \|\theta_Z(r)\|_m^2 dr &= \|G\|_m^2 + 2 \int_s^T \langle \theta_Z \theta_Y(r), \theta_Y(r) \rangle_m dr \\ &\leq \|G\|_m^2 + C \int_s^T \|\theta_Z(r)\|_{L^\infty(\mathbb{R}^2)} \|\theta_Y(r)\|_m^2 dr \end{aligned} \quad (5.6)$$

which by Gronwall inequality implies

$$\|\theta_Y(s)\|_m \leq C\|G\|_m \exp \left(\int_s^T \|\theta_Z(r)\|_{L^\infty(\mathbb{R}^2)} dr \right).$$

In view of (5.5), we have

$$\ln^+ \|\theta_Y(s)\|_m \leq C(\|G\|_m, T) \left(1 + \int_s^T \ln^+ \|\theta_Y(r)\|_m dr \right).$$

Gronwall inequality yields that

$$\sup_{s \in [t, T]} \|\theta_Y(s)\|_m \leq C(\|G\|_m, T), \quad \forall t \in (T_0, T]. \quad (5.7)$$

using a bootstrap argument, we can extend the local H^m -solution (X, Y, Z, \tilde{Y}_0) of FBSDS (1.3) into a global one. Therefore, we have

Theorem 5.2. *Let $d = 2$, $m \geq 3$ and $G \in H_\sigma^m$. Then our FBSDS (1.3) with $f = 0$ admits a unique H^m -solution (X, Y, Z, \tilde{Y}_0) .*

6 Appendix

6.1 Proof of Lemma 3.1

It is sufficient for us to prove (3.3) with $l = 1$, from which (3.4) follows by Fubini Theorem.

First, taking a nonnegative function $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$, we consider the following trivial FBSDE:

$$\begin{cases} dX_r(t, x) = b(r, X_r(t, x)) dr + \sqrt{\nu} dW_r, & T - \varepsilon \leq t \leq r \leq s; \\ X_t(t, x) = x; \\ dY_r(t, x) = \sqrt{\nu} Z_r(t, x) dW_r, & r \in [t, s]; \\ Y_T(t, x) = \varphi(X_s(t, x)). \end{cases} \quad (6.1)$$

In view of Lemma 4.2 and the proof therein, FBSDE (6.1) is a particular case with $\phi = 0$ therein, and moreover, the assertions of Lemma 4.2 still hold for (6.1), as Lemma 3.1 will never be involved in the proof of Lemma 4.2 if $\phi = 0$. Therefore, for almost all $x \in \mathbb{R}^d$ our FBSDE (6.1) admits a unique solution

$$(X_r(t, x), Y_r(t, x), Z_r(t, x)) \in S^2(t, s; \mathbb{R}^d) \times S^2(t, s; \mathbb{R}^d) \times L^2_{\mathcal{F}}(t, s; \mathbb{R}^d),$$

and for this solution (X, Y, Z) , there hold

$$\begin{aligned} \theta_Y &\in C([t, s]; H^m) \cap L^2(t, s; H^{m+1}) \\ \theta_Y(r, X_r(t, x)) &= \varphi(X_s(t, x)) - \sqrt{\nu} \int_r^s \theta_Z(\tau, X_\tau(t, x)) dW_\tau, \text{ a.s.} \end{aligned}$$

and

$$\theta_Z(t, x) = \nabla \theta_Y(t, x), \quad Y_r(t, x) = \theta_Y(r, X_r(t, x)), \quad Z_r(t, x) = \theta_Z(r, X_r(t, x)), \text{ a.s., } t \leq r \leq s. \quad (6.2)$$

In an obvious way, we have

$$Y_r(t, x) = E[\varphi(X_s(t, x)) | \mathcal{F}_r] \geq 0, \quad \text{a.s. } r \in [t, s].$$

Define the following equivalent probability measure

$$d\mathbb{Q}^{t,x} =: \exp\left(-\nu^{-\frac{1}{2}} \int_t^s b(r, X_r(t, x)) dW_s - \frac{1}{2} \nu^{-1} \int_t^s |b(r, X_r(t, x))|^2 ds\right) d\mathbb{P}.$$

In view of (6.2), FBSDE (6.1) reads

$$\left\{ \begin{array}{l} dX_r(t, x) = \sqrt{\nu} dW'_r, \quad t \leq r \leq s; \\ X_t(t, x) = x; \\ -dY_r(t, x) = Z_r(t, x) b(r, X_r(t, x)) dr - \sqrt{\nu} Z_r(t, x) dW'_r \\ \quad = (b \cdot \nabla) \theta_Y(r, X_r(t, x)) dr - \sqrt{\nu} Z_r(t, x) dW'_r, \quad r \in [t, s]; \\ Y_s(t, x) = \varphi(X_s(t, x)), \end{array} \right. \quad (6.3)$$

where $(W', \mathbb{Q}^{t,x})$ is a standard Brownian motion. Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^d} \theta_Y(r, x) dx \\ &= \int_{\mathbb{R}^d} E_{\mathbb{Q}^{t,x}}[\theta_Y(r, X_r(t, x))] dx \\ &= \int_{\mathbb{R}^d} E_{\mathbb{Q}^{t,x}}[\varphi(X_s(t, x))] dx + \int_{\mathbb{R}^d} \int_r^s E_{\mathbb{Q}^{t,x}}[(b \cdot \nabla) \theta_Y(\tau, X_\tau(t, x))] d\tau dx \\ &= \int_{\mathbb{R}^d} \varphi(x) dx + \int_{\mathbb{R}^d} \int_r^s (b \cdot \nabla) \theta_Y(\tau, x) d\tau dx \\ &= \int_{\mathbb{R}^d} \varphi(x) dx - \int_{\mathbb{R}^d} \int_r^s (\operatorname{div} b) \theta_Y(\tau, x) d\tau dx \\ &\leq \int_{\mathbb{R}^d} \varphi(x) dx + \int_r^s \|\operatorname{div} b(\tau)\|_{L^\infty} \int_{\mathbb{R}^d} \theta_Y(\tau, x) dx d\tau, \\ &\text{or} \\ &\geq \int_{\mathbb{R}^d} \varphi(x) dx - \int_r^s \|\operatorname{div} b(\tau)\|_{L^\infty} \int_{\mathbb{R}^d} \theta_Y(\tau, x) dx d\tau. \end{aligned}$$

Using Gronwall inequality, we have

$$\kappa \int_{\mathbb{R}^d} \varphi(x) dx \leq \int_{\mathbb{R}^d} \theta_Y(r, x) dx \leq \kappa^{-1} \int_{\mathbb{R}^d} \varphi(x) dx, \quad \forall r \in [t, s]$$

with

$$\kappa := e^{-\|\operatorname{div} b\|_{L^1(0,T;L^\infty)}}.$$

Taking $r = t$, we have (3.3) for $K := \kappa^{-1}$.

For the general function $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$ without the nonnegative assumption, we choose a positive Schwartz function h and a nonnegative function $\tilde{\varphi} \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$ such that

$$\operatorname{supp} \varphi \subset \{x \in \mathbb{R}^d : \tilde{\varphi}(x) = 1\}.$$

Set

$$\varphi_\varepsilon := \sqrt{\varphi^2 + \varepsilon h} \tilde{\varphi}, \text{ for } \varepsilon \in (0, 1).$$

Then in view of the above arguments, we have

$$\kappa \|\varphi\|_{L^1(\mathbb{R}^d)} \leq \kappa \|\varphi_\varepsilon\|_{L^1(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} E[|\varphi_\varepsilon(X_s(t, x))|] dx \leq K \|\varphi_\varepsilon\|_{L^1(\mathbb{R}^d)}.$$

Letting $\varepsilon \rightarrow 0$, we conclude from Lebesgue dominant convergence theorem that (3.3) holds for all $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$.

Finally, for any $\varphi \in L^1(\mathbb{R})$, we choose a sequence $\{\varphi^n, n \in \mathbb{Z}^+\} \subset C_c^\infty(\mathbb{R}^d; \mathbb{R})$ such that $\lim_{n \rightarrow \infty} \|\varphi - \varphi^n\|_{L^1(\mathbb{R})} = 0$. Then, by (3.3), $\{\varphi^n(X_s(t, x))\}$ is a Cauchy sequence in $L^1(\Omega \times \mathbb{R}^d; \mathbb{R})$. It remains to show that $\varphi(X_s(t, \cdot))$ is the limit.

Through the above approximation, we can check that (3.3) holds for any continuous function of a compact support. Therefore, if $A \subset \mathbb{R}^d$ is a measurable, bounded subset of zero Lebesgue measure, then the $d\mathbb{P} \times dx$ -measure of the set $\{(\omega, x) \in \Omega \times \mathbb{R}^d : X_s(t, x) \in A\}$ is zero. Thus, the almost everywhere convergence of φ^n to φ in \mathbb{R}^d implies that of $\varphi^n(X_s(t, \cdot))$ to $\varphi(X_s(t, \cdot))$.

Hence, $\varphi^n(X_s(t, x))$ converges to $\varphi(X_s(t, x))$ in $L^1(\Omega \times \mathbb{R}^d; d\mathbb{P} \times dx)$. Since (3.3) holds for each φ^n , passing to the limit, (3.3) holds for any $\varphi \in L^1(\mathbb{R})$. We complete the proof.

6.2 Proof of Lemma 4.1

Step 1. For the trivial case of $\alpha = 0$, the proof is referred to [36, 37] and it becomes a special one in [1, 20, 29, 35]. It is sufficient to consider the case of $\alpha \neq 0$.

Step 2. We prove the existence of the solution (X, Y, Z) which satisfies all the assertions of Lemma 4.1.

Choosing two positive real numbers $\varepsilon < T$ and M whose values are to be determined later, define

$$V_M^\varepsilon := \left\{ u \in C([T - \varepsilon, T]; H_\sigma^m) \cap L^2(T - \varepsilon, T; H_\sigma^{m+1}) : \right. \\ \left. \|u\|_{C([T - \varepsilon, T]; H_\sigma^m)}^2 + \frac{\nu}{2} \|\nabla u\|_{L^2(T - \varepsilon, T; H_\sigma^{m+1})}^2 \leq M^2 \right\}.$$

For each $\zeta \in V_M^\varepsilon$, through a similar and simpler approximating method to be used in Proposition 4.3 we prove that the following FBSDE:

$$\begin{cases} dX_s(t, x) = [b(s, X_s(t, x)) + \alpha \zeta(s, X_s(t, x))] ds + \sqrt{\nu} dW_s, & T - \varepsilon \leq t \leq s \leq T \\ X_t(t, x) = x; \\ -dY_s(t, x) = \phi(s, X_s(t, x)) ds - \sqrt{\nu} Z_s(t, x) dW_s, & s \in [t, T] \\ Y_T(t, x) = \psi(X_T(t, x)) \end{cases} \quad (6.4)$$

admits a unique solution triple $(X^\zeta, Y^\zeta, Z^\zeta)$ which satisfies

$$\begin{aligned}
& \|\theta_{Y^\zeta}(s)\|_m^2 + \nu \int_s^T \|\theta_{Z^\zeta}(r)\|_m^2 dr \\
&= \|\psi\|_m^2 + \int_s^T 2\langle (b + \alpha\zeta) \cdot \nabla \theta_{Y^\zeta}(r), \theta_{Y^\zeta}(r) \rangle_{m-1, m+1} dr + \int_s^T 2\langle \phi(r), \theta_{Y^\zeta}(r) \rangle_{m-1, m+1} dr \\
&\leq \|\psi\|_m^2 + C \left(\int_s^T \|b(r) + \alpha\zeta(r)\|_m^2 \|\theta_{Y^\zeta}(r)\|_m^2 dr + \int_s^T \|\phi(r)\|_{m-1}^2 dr \right) \\
&\quad + \frac{\nu}{2} \int_s^T (\|\theta_{Y^\zeta}(r)\|_m^2 + \|\theta_{Z^\zeta}(r)\|_m^2) dr,
\end{aligned} \tag{6.5}$$

where we have used Remark 2.1. By Gronwall inequality, we have

$$\begin{aligned}
& \sup_{s \in [T-\varepsilon, T]} \|\theta_{Y^\zeta}(s)\|_m^2 + \frac{\nu}{2} \int_{T-\varepsilon}^T \|\theta_{Z^\zeta}(r)\|_m^2 dr \\
&\leq C \left(\|\phi\|_{L^2(0, T; H^{m-1})}^2 + \|\psi\|_m^2 \right) e^{\left(\|b\|_{C([T-\varepsilon, T]; H^m)} + |\alpha| \|\zeta\|_{C([T-\varepsilon, T]; H^m)} \right)^2 \varepsilon}.
\end{aligned} \tag{6.6}$$

Choosing M to be big enough and ε to be small enough, we obtain

$$\sup_{s \in [T-\varepsilon, T]} \|\theta_{Y^\zeta}(s)\|_m^2 + \frac{\nu}{2} \int_{T-\varepsilon}^T \|\theta_{Z^\zeta}(r)\|_m^2 dr \leq M^2.$$

On the other hand, for any $\zeta_1, \zeta_2 \in V_M^\varepsilon$, setting

$$(\delta\theta_{Y^\zeta}, \delta\theta_{Z^\zeta}, \delta\zeta) := (\theta_{Y^{\zeta_1}} - \theta_{Y^{\zeta_2}}, \theta_{Z^{\zeta_1}} - \theta_{Z^{\zeta_2}}, \zeta_1 - \zeta_2),$$

we have

$$\begin{aligned}
& \|\delta\theta_{Y^\zeta}(s)\|_m^2 + \nu \int_s^T \|\delta\theta_{Z^\zeta}(r)\|_m^2 dr \\
&= 2 \int_s^T \alpha \langle (\delta\zeta \cdot \nabla) \theta_{Y^{\zeta_1}}(r), \delta\theta_{Y^\zeta}(r) \rangle_{m-1, m+1} dr + 2 \int_s^T \alpha \langle (\zeta_2 \cdot \nabla) \delta\theta_{Y^\zeta}(r), \delta\theta_{Y^\zeta}(r) \rangle_{m-1, m+1} dr \\
&\leq \frac{\nu}{2} \int_s^T (\|\delta\theta_{Y^\zeta}(r)\|_m^2 + \|\delta\theta_{Z^\zeta}(r)\|_m^2) dr \\
&\quad + C \left(\int_s^T \|\delta\zeta(r)\|_m^2 \|\theta_{Y^{\zeta_1}}(r)\|_m^2 dr + \int_s^T \|\zeta_2(r)\|_m^2 \|\delta\theta_{Y^\zeta}(r)\|_m^2 dr \right)
\end{aligned}$$

which together with the Gronwall-Bellman inequality, implies

$$\sup_{s \in [T-\varepsilon, T]} \|\delta\theta_{Y^\zeta}(s)\|_m^2 + \frac{\nu}{2} \int_{T-\varepsilon}^T \|\delta\theta_{Z^\zeta}(r)\|_m^2 dr \leq CM^2 e^{M^2 T} \varepsilon \|\delta\zeta\|_{C([T-\varepsilon, T]; H_\sigma^m)}^2.$$

Therefore, if we choose ε to be small enough, the solution map $\Psi : \zeta \mapsto \theta_{Y^\zeta}$ is a contraction mapping on the complete metric space V_M^ε and then through bootstrap arguments, we obtain a unique function $\bar{\zeta} \in C_{\text{loc}}((\tau, T]; H_\sigma^m) \cap L_{\text{loc}}^2(\tau, T; H_\sigma^{m+1})$ satisfying $(\theta_{Y^{\bar{\zeta}}}, \theta_{Z^{\bar{\zeta}}}) = (\bar{\zeta}, \nabla \bar{\zeta})$ on $(\tau, T] \times \mathbb{R}^d$. where $T - \tau$ continuously depends on $\|\phi\|_{L^2(0, T; H^{m-1})}, \|b\|_{C([0, T]; H^m)}, \|\psi\|_m, \nu$ and α . Furthermore, By Lemma 4.2, solving FBSDE (6.4) with ζ replaced by $\bar{\zeta}$, we get $(X^{\bar{\zeta}}, Y^{\bar{\zeta}}, Z^{\bar{\zeta}})$ which is a solution to our FBSDE (4.1) and satisfies (4.2), (4.3) and (4.4).

Step 3. In a similar way to **Step 2** of the proof of Proposition 4.3, we prove the uniqueness. We complete the proof.

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